Problem 14 in Section 3.2. The problem tells us that $y_1 = e^x$, $y_2 = e^{2x}$, and $y_3 = e^{3x}$ all are solutions of the Differential equation $y''' - 6y'' + 11y' - 6y = 0$. We are supposed to solve the Initial Value Problem

$$y''' - 6y'' + 11y' - 6y = 0, \quad y(0) = 0, \quad y'(0) = 0, \quad y''(0) = 3.$$

Solution. The general solution of $y''' - 6y'' + 11y' - 6y = 0$ is

$$y = c_1 e^x + c_2 e^{2x} + c_3 e^{3x}.$$

Our job is to evaluate the constants. We calculate

$$\begin{aligned} y' &= c_1 e^x + 2c_2 e^{2x} + 3c_3 e^{3x} \\ y'' &= c_1 e^x + 4c_2 e^{2x} + 9c_3 e^{3x} \end{aligned}$$

We must solve

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= c_1 + 2c_2 + 3c_3 \\ 3 &= c_1 + 4c_2 + 9c_3 \end{aligned}$$

Replace Equation 2 with Equation 2 minus Equation 1.
Replace Equation 3 with Equation 3 minus Equation 1.

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= \quad\ \ 1c_2 + 2c_3 \\ 3 &= \quad\ \ 3c_2 + 8c_3 \end{aligned}$$

Replace Equation 3 with Equation 3 minus 3 times Equation 2.

$$\begin{aligned} 0 &= c_1 + c_2 + c_3 \\ 0 &= \quad\ \ 1c_2 + 2c_3 \\ 3 &= \qquad\qquad 2c_3 \end{aligned}$$

We see that $c_3 = \frac{3}{2}$, $c_2 = -3$, and $c_1 = \frac{3}{2}$. We conclude that the solution of the Initial Value Problem is

$$\boxed{y = \tfrac{3}{2}e^x - 3e^{2x} + \tfrac{3}{2}e^{3x}.}$$

Check. We compute

$$\begin{aligned} y &= \frac{3}{2}e^{x} - 3e^{2x} + \frac{3}{2}e^{3x} \\ y' &= \frac{3}{2}e^{x} - 6e^{2x} + \frac{9}{2}e^{3x} \\ y'' &= \frac{3}{2}e^{x} - 12e^{2x} + \frac{27}{2}e^{3x} \\ y''' &= \frac{3}{2}e^{x} - 24e^{2x} + \frac{81}{2}e^{3x} \end{aligned}$$

It follows that

$$y''' - 6y'' + 11y' - 6y = \begin{cases} \frac{3}{2}e^{x} - 24e^{2x} + \frac{81}{2}e^{3x} \\ -6(\frac{3}{2}e^{x} - 12e^{2x} + \frac{27}{2}e^{3x}) \\ +11(\frac{3}{2}e^{x} - 6e^{2x} + \frac{9}{2}e^{3x}) \\ -6(\frac{3}{2}e^{x} - 3e^{2x} + \frac{3}{2}e^{3x}) \end{cases}$$

$$= (\tfrac{3}{2} - 9 + \tfrac{33}{2} - 9)e^{x} + (-24 + 72 - 66 + 18)e^{2x} + \underbrace{\left(\tfrac{81}{2} - 6(\tfrac{27}{2}) + 11(\tfrac{9}{2}) - 6(\tfrac{3}{2})\right)}_{\frac{3}{2}(27-54+33-6)=0} e^{3x} = 0,$$

as expected.✓We also see that

$$y(0) = \tfrac{3}{2}e^{0} - 3e^{0} + \tfrac{3}{2}e^{0} = (\tfrac{3}{2} - 3 + \tfrac{3}{2}) = 0; \checkmark$$

$$y'(0) = \tfrac{3}{2}e^{0} - 6e^{0} + \tfrac{9}{2}e^{0} = (\tfrac{3}{2} - 6 + \tfrac{9}{2}) = 0; \checkmark$$

and

$$y''(0) = \tfrac{3}{2}e^{0} - 12e^{0} + \tfrac{27}{2}e^{0} = \tfrac{3}{2} - 12 + \tfrac{27}{2} = 15 - 12 = 3. \checkmark$$

Our proposed solution does everything it is supposed to do. It is correct.