Problem 1 in Section 2.4. Use Euler's method to approximate $y(\frac{1}{2})$ where y is a solution of the Initial Value Problem

$$y' = -y \quad \text{and} \quad y(0) = 2.$$

Use two steps only; in other words, take $h = \frac{1}{4}$. Compare your approximation of $y(\frac{1}{2})$ to the actual value of $y(\frac{1}{2})$.

Solution. We put the picture on the next page. The number y_2 is our approximation of $y(\frac{1}{2})$.

We first find y_1. The line segment from $(0, 2)$ to $(\frac{1}{4}, y_1)$ has slope equal to $m_1 = -2$:

$$\frac{y_1 - 2}{\frac{1}{4}} = -2, \quad y_1 = 2 - \frac{1}{2} = \frac{3}{2}.$$

Now we find y_2. The line segment from $(\frac{1}{4}, \frac{3}{2})$ to $(\frac{1}{2}, y_2)$ has slope equal to $m_2 = -y_1 = -\frac{3}{2}$:

$$\frac{y_2 - \frac{3}{2}}{\frac{1}{4}} = -\frac{3}{2}, \quad y_2 = \frac{3}{2} - \frac{3}{2}\frac{1}{4} = \frac{12}{8} - \frac{3}{8} = \frac{9}{8}.$$

$$\boxed{\text{Euler's Method, with } n = 2\text{, gives } \frac{9}{8} \text{ as the approximation of } y(\frac{1}{2}).}$$

It turns out that we can find the real value of $y(\frac{1}{2})$. To solve

$$\frac{dy}{dx} = -y,$$

separate the variables (Get all y's and dy's on one side and all x's and dx's on the other side.) then integrate.

$$\frac{dy}{y} = -dx$$

$$\int \frac{dy}{y} = -\int dx$$

$$\ln y = -x + C$$

Exponentiate both sides:

$$e^{\ln y} = e^C e^{-x}$$

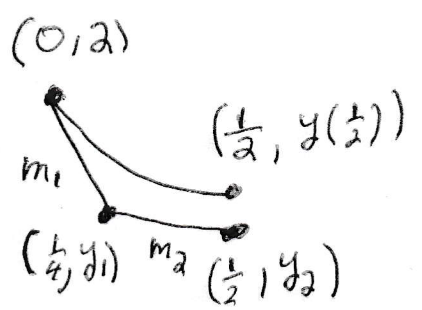
$$y = e^C e^{-x}.$$

Plug in the initial condition $y(0) = 2$ to see that

$$2 = e^C.$$

Conclude that $y = 2e^{-x}$ is the solution of the Initial Value Problem. So, $y(\frac{1}{2})$ is equal to $\frac{2}{\sqrt{e}}$. If you are keeping score, the beginning of the decimal expansion for $\frac{2}{\sqrt{e}}$ is 1.2130613 and the decimal expansion of our approximation $\frac{9}{8}$ is 1.125.

Picture for 2.4 Number 1



The smooth curve is the real solution of the initial Value Problem $y' = -y \quad y(0)=2 \quad (*)$

The piece-wise linear curve is the Euler Method Approximation of the solution of $(*)$ made using two steps.

m_1 is the slope of the line segment from $(0,2)$ to $(\frac{1}{4}, y_1)$

m_2 is the slope of the line segment from $(\frac{1}{4}, y_1)$ to $(\frac{1}{2}, y_2)$

We make $m_1 = f(0,2)$ and $m_2 = f(\frac{1}{4}, y_1)$ where $f(x,y)$ is the right side of the Differential Equation in $(*)$

In other words $f(x,y) = -y$

$m_1 = f(0,2) = -2$

$m_2 = f(\frac{1}{4}, y_1) = -y_1$