Problem 7 in Section 2.2. Consider the Differential Equation $\frac{dx}{dt} = (x-2)^2$.

- What are the equilibrium solutions of this autonomous Differential Equation?
- Draw the phase diagram for this Differential Equation.
- Are the equilibrium solutions of this Differential Equation stable or unstable?
- Sketch the solution of the Initial Value problem
$$\frac{dx}{dt} = (x-2)^2 \quad x(0) = x_0, \tag{9}$$
for a few choices of x_0.
- Solve the Initial Value Problem (9). (Be sure that your answer is in the form x is equal to some function of t.)

We see that $(x-2)^2$ is equal to zero when $x = 2$. We conclude that $x = 2$ is the only equilibrium solution of the given Differential Equation. We see that $(x-2)^2$ is positive for all x other than $x = 2$. (The phase diagram is on the last page of this solution.) We conclude that

- if $2 < x_0$, then the graph of $x = x(t)$ increases away from 2 as t goes to infinity; and
- if $x_0 < 2$, then the graph of $x = x(t)$ increases towards to 2 as t goes to infinity.

In other words,

The function $x = 2$ is a semi-stable equilibrium solution of $\frac{dx}{dt} = (x-2)^2$. It is stable from below, but unstable from above.

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On the last page of this solution, we sketched three solutions of the Initial Value Problem
$$\frac{dx}{dt} = (x-2)^2 \quad x(0) = x_0.$$
In one solution x_0 is bigger than 2; in one solution x_0 is equal to 2; and in one solution x_0 is less than 2

We solve $\frac{dx}{dt} = (x-2)^2$ by separating the variables and integrating:

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We solve $\frac{dx}{dt} = (x-2)^2$ by separating the variables and integrating:

$$\int \frac{dx}{(x-2)^2} = \int dt$$

$$-(x-2)^{-1} = t + C$$

Set $t = 0$ to determine C:

$$-\frac{1}{x_0 - 2} = C. \tag{10}$$

We continue to write C. We will insert (10) into our answer later.

$$\frac{-1}{t+C} = x - 2$$

$$2 - \frac{1}{t+C} = x$$

Now insert (10):

$$2 - \frac{1}{t - \frac{1}{x_0 - 2}} = x$$

Multiply top and bottom by $x_0 - 2$:

$$\boxed{2 - \frac{x_0 - 2}{(x_0 - 2)t - 1} = x}$$

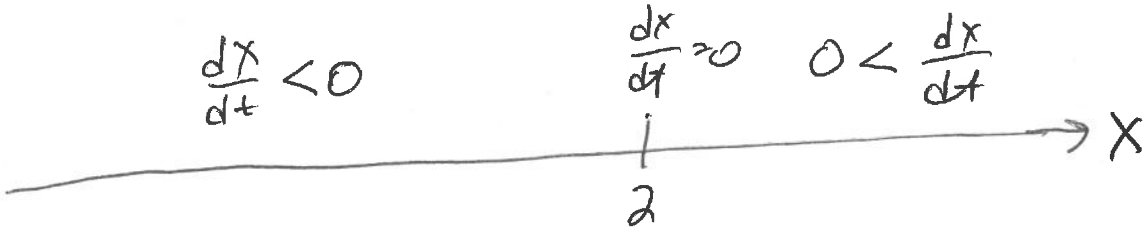
Check: We plug our proposed answer into $\frac{dx}{dt} - (x-2)^2$ and obtain

$$\frac{(x-2)^2}{((x_0-2)t-1)^2} - \left(-\frac{x_0 - 2}{(x_0-2)t-1}\right)^2 = 0.$$

Our proposed answer is correct.

The pictures are on the next page.

The pictures for Problem 7 in Section 2.2



A few solutions of $\frac{dx}{dt} = (x-2)^2 \quad x(0) = x_0$