Problem 3 in Section 2.2. Consider the Differential Equation $\frac{dx}{dt} = x^2 - 4x$.

- What are the equilibrium solutions of this autonomous Differential Equation?
- Draw the phase diagram for this Differential Equation.
- Are the equilibrium solutions of this Differential Equation stable or unstable?
- Sketch the solution of the Initial Value problem

$$\frac{dx}{dt} = x^2 - 4x \quad x(0) = x_0, \tag{4}$$

  for a few choices of x_0.

- Solve the Initial Value Problem (4). (Be sure that your answer is in the form x is equal to some function of t.)

We see that $x^2 - 4x = x(x-4)$ and this expression is equal to 0 when $x = 0$ or $x = 4$. Thus $x = 0$ and $x = 4$ are the equilibrium solutions of $\frac{dx}{dt} = x^2 - 4x$.

We also see that

- if $4 < x$, then $x(x-4)$ is positive;
- if $0 < x < 4$, then $x(x-4)$ is negative; and
- if $x < 0$, then $x(x-4)$ is positive.

(We drew a Phase Diagram for the Differential Equation $\frac{dx}{dt} = x^2 - 4x$ on the last page of this solution.)

In particular,

- if $4 < x(0)$, then the graph of $x = x(t)$ increases away from 4 as t goes to infinity;
- if $0 < x(0) < 4$, then the graph of $x = x(t)$ decreases away from 4 and towards 0 as t goes to infinity; and
- if $x(0) < 0$, then the graph of $x = x(t)$ increases toward 0 as t goes to infinity.

So $x = 4$ is an unstable equilibrium and $x = 0$ is a stable equilibrium for the Differential Equation $\frac{dx}{dt} = x^2 - 4x$.

On the last page of this solution, we sketched five solutions of the Initial Value Problem

$$\frac{dx}{dt} = x^2 - 4x \quad x(0) = x_0.$$

In one solution x_0 is bigger than 4; in one solution x_0 is equal to 4; in one solution x_0 is between 0 and 4; in one solution x_0 is equal to 0; and in one solution x_0 is less than zero.

We solve $\frac{dx}{dt} = x^2 - 4x$ by separating the variables and integrating:

$$\int \frac{dx}{x^2 - 4x} = \int dt. \tag{5}$$

Factor the denominator and use the technique of Partial Fractions:

$$\tfrac{1}{x^2-4x} = \tfrac{1}{x(x-4)} = \tfrac{A}{x} + \tfrac{B}{x-4}.$$

Multiply both sides of $\frac{1}{x(x-4)} = \frac{A}{x} + \frac{B}{x-4}$ by $x(x-4)$ to obtain

$$1 = A(x-4) + Bx.$$

Plug in $x = 0$ to learn that $A = \frac{-1}{4}$. Plug in $x = 4$ to learn that $B = \frac{1}{4}$. It follows that

$$\tfrac{1}{4}(\tfrac{1}{x-4} - \tfrac{1}{x}) = \tfrac{1}{x(x-4)}.$$

The equation (5) becomes

$$\tfrac{1}{4}\int(\tfrac{1}{x-4} - \tfrac{1}{x})dx = t + C$$

$$\tfrac{1}{4}(\ln|x-4| - \ln|x|) = t + C$$

$$(\ln|x-4| - \ln|x|) = 4t + 4C$$

Exponentiate:

$$|\tfrac{x-4}{x}| = e^{4C}e^{4t}$$

$$\tfrac{x-4}{x} = \pm e^{4C}e^{4t}$$

Let K be the constant $\pm e^{4C}$.

$$\tfrac{x-4}{x} = Ke^{4t}$$

Multiply both sides by x:

$$x - 4 = Ke^{4t}x$$

Subtract $Ke^{4t}x$ from each side and add 4 to each side:

$$x - Ke^{4t}x = 4$$

$$x(1 - Ke^{4t}) = 4$$

$$x = \tfrac{4}{1-Ke^{4t}}$$

Plug in $t = 0$ to find K:

$$x_0 = \tfrac{4}{1-K}$$

$$1 - K = \tfrac{4}{x_0}$$

$$1 - \frac{4}{x_0} = K$$

$$\frac{x_0 - 4}{x_0} = K$$

Thus the solution of the Initial Value Problem (4) is

$$x = \frac{4}{1 - (\frac{x_0 - 4}{x_0})e^{4t}}$$

or

$$x = \frac{4x_0}{x_0 - (x_0 - 4)e^{4t}}$$

or

$$\boxed{x = \frac{4x_0}{x_0 + (4 - x_0)e^{4t}}}$$

Check. We plug our candidate for x into

$$\frac{dx}{dt} - x^2 + 4x$$

and then clean the thing up. We hope that it simplifies to become zero. When we take the derivative of x, we view x as a constant times u^{-1}, for some function u of t. The derivative of u^{-1} is $-u^{-2}\frac{du}{dt}$.

At any rate when we plug $x = \frac{4x_0}{x_0 + (4 - x_0)e^{4t}}$ into

$$\frac{dx}{dt} - x^2 + 4x,$$

we obtain

$$\frac{-4x_0(4 - x_0)4e^{4t}}{(x_0 + (4 - x_0)e^{4t})^2} - \frac{(4x_0)^2}{(x_0 + (4 - x_0)e^{4t})^2} + \frac{16x_0}{x_0 + (4 - x_0)e^{4t}}$$

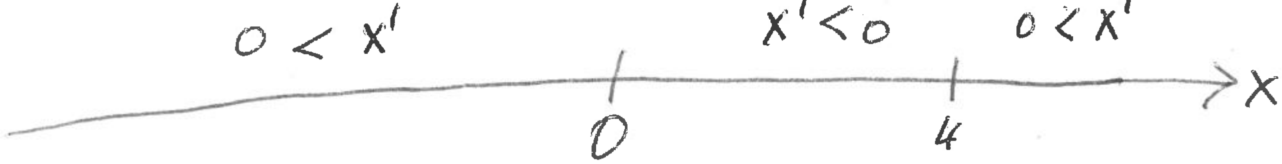
$$= \frac{1}{(x_0 + (4 - x_0)e^{4t})^2}\Big(- 4x_0(4 - x_0)4e^{4t} - (4x_0)^2 + 16x_0(x_0 + (4 - x_0)e^{4t})\Big),$$

and this is zero. Our proposed answer is correct.

The pictures are on the next page.

Picture for HW 2.2 number 3

Phase Diagram for $x' = x^2 - 4x$



5 solutions of $x' = x^2 - 4x \quad x(0) = x_0$