Problem 33 in Section 2.1 Solve the Initial Value Problem

$$\frac{dP}{dt} = kP(P-M) \quad P(0) = P_0.$$

Draw the solution for

- $P_0 = M$,
- $M < P_0$, and
- $P_0 < M$.

Solution.

We solve

$$\frac{dP}{dt} = kP(P-M),$$

with k and M positive. We separate the variables and do the partial fractions:

$$\frac{dP}{P(P-M)} = kdt,$$

$$\int \frac{1}{M}\left(\frac{1}{P-M} - \frac{1}{P}\right) dP = \int kdt,$$

$$\ln|P-M| - \ln P = Mkt + C,$$

$$\frac{|P-M|}{P} = e^C e^{Mkt},$$

$$\frac{P-M}{P} = Ke^{Mkt},$$

where $K = \pm e^C$,

$$P - M = Ke^{Mkt}P.$$

At this point we calculate that $\frac{P(0)-M}{P(0)} = K$. Move all the terms with P to the left and all of the terms without P to the right:

$$P(1 - Ke^{Mkt}) = M,$$

$$P(t) = \frac{M}{1 - Ke^{Mkt}},$$

$$P(t) = \frac{M}{1 - \frac{(P(0)-M)}{P(0)}e^{Mkt}},$$

$$P(t) = \frac{P(0)M}{P(0) - (P(0)-M)e^{Mkt}}.$$

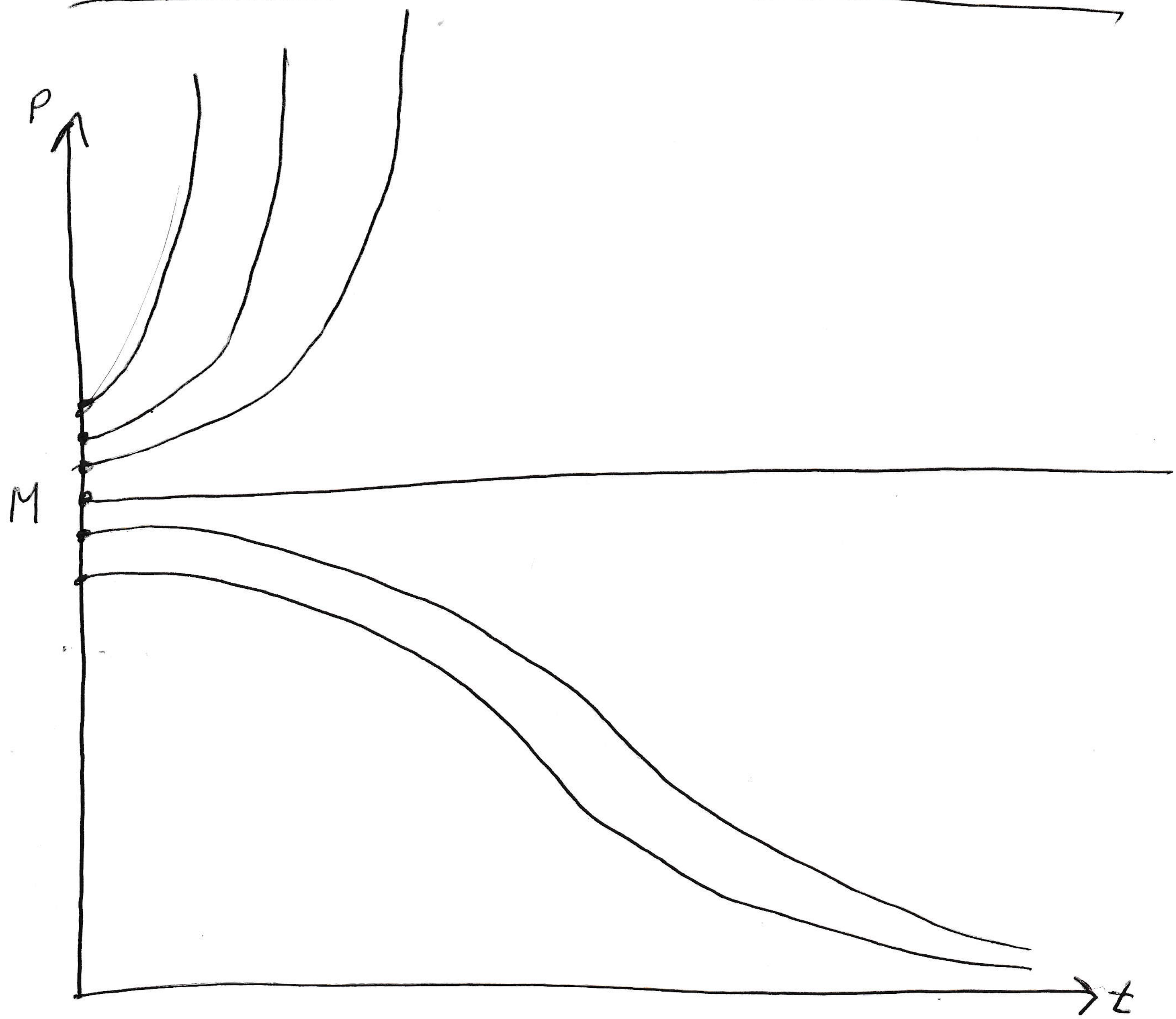
We draw some conclusions.

- If $P(0) = M$, then $P(t) = M$ for all t.
- If $P(0) < M$, then the denominator is always positive and goes to $+\infty$ as t goes to ∞. Thus, in this case $\lim_{t\to\infty} P(t) = 0$. (In this case the population becomes extinct!)
- If $M < P(0)$, then the denominator starts positive but eventually goes to $-\infty$ Thus, there is a finite time when the denominator becomes 0. In other words, there is a finite time when the population $P(t)$ **explodes** to $+\infty$. (In problem 45 in section 1.1 a population of rats exploded.)

The DE $\frac{dP}{dt} = kP(P - M)$ is called the extinction/explosion DE because there is a magic threshold. If the initial population is less than the threshold, then the population will die out. If the initial population is above the threshold, then the population will grow out of control. Neither of these situations is sustainable!

The graph of $P(t)$ for various choices of $P(0)$ looks like:

Solutions of the Extinction/Explosion Equation



$$\frac{dP}{dt} = kP(P-M) \qquad \text{with } 0 < k, M$$