Problem 32 in Section 2.1 Solve the Initial Value Problem

$$\tfrac{dP}{dt} = kP(M-P) \quad P(0) = P_0.$$

Draw the solution for

- $P_0 = M$,
- $M < P_0$, and
- $P_0 < M$.

Solution.

We solve

$$\frac{dP}{dt} = kMP - kP^2.$$

We can separate the variables:

$$\frac{dP}{MP - P^2} = k\,dt. \tag{1}$$

It is not difficult to integrate

$$\frac{1}{P(M-P)}.$$

We use the technique of partial fractions and see what $\frac{1}{P(M-P)}$ used to look like before some body "cleaned it up". It used to be

$$\frac{1}{P(M-P)} = \frac{A}{P} + \frac{B}{M-P}$$

for some numbers A and B. We can figure out A and B. Clear the denominators:

$$1 = A(M-P) + BP.$$

(The last equation holds for all P. The number M is fixed and not zero. Our job is to find A and B.) Plug in $P = M$ to learn that $\frac{1}{M} = B$. Plug in $P = 0$ to learn that $\frac{1}{M} = A$. Observe that

$$\frac{1}{M}\left(\frac{1}{P} + \frac{1}{M-P}\right)$$

really does equal $\frac{1}{P(M-P)}$. Integrate Equation (1) to obtain

$$\frac{1}{M}\int\left(\frac{1}{P} + \frac{1}{M-P}\right)dP = \int kdt$$

and

$$\frac{1}{M}(\ln P - \ln|M-P|) = kt + C.$$

(There is no need to write $|P|$, because P is a population; hence it can not be negative.) Multiply both sides of the equation by M to get

$$\ln\left(\frac{P}{|M-P|}\right) = Mkt + MC$$

Exponentiate to obtain

$$\frac{P}{|M-P|} = e^{MC}e^{Mkt}.$$

Of course, $|M-P| = \pm(M-P)$. Move $\pm$ to the other side and let $K = \pm e^{MC}$.

$$\frac{P}{M-P} = Ke^{Mkt}. \tag{2}$$

This is a good time to calculate K. Plug $t = 0$ into both sides of (2) to learn that

$$\frac{P(0)}{M-P(0)} = K. \tag{3}$$

We want to solve for P; so we multiply both sides of (2) by $M - P$ to obtain

$$P = Ke^{Mkt}(M-P).$$

Add $Ke^{Mkt}P$ to both sides

$$P(1+Ke^{Mkt}) = Ke^{Mkt}M.$$

Divide both sides by $1 + Ke^{Mkt}$ and obtain

$$P = \frac{Ke^{Mkt}M}{1+Ke^{Mkt}}.$$

This is **a** formula for $P(t)$; but lets clean it up a little! Instead of having two competing exponential functions, we arrange things so that there is only one exponential function. Instead of having the constant K appear twice, we arrange things so K only appears once. Divide top and bottom by Ke^{Mkt}. Thus,

$$P = \frac{M}{\frac{e^{-Mkt}}{K}+1}.$$

Replace K by the value given in (3).

$$P = \frac{M}{\frac{(M-P(0))e^{-Mkt}}{P(0)}+1}.$$

Multiply top and bottom by $P(0)$. We have calculated that

$$P(t) = \frac{MP(0)}{(M-P(0))e^{-Mkt}+P(0)},$$

or

$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{-Mkt}}.$$

Here is the first cool observation. If the population $P(t)$ is governed by the logistic equation, then the population is sustainable. That is,

$$\lim_{t\to\infty} P(t)$$

exists and is finite. In particular,

$$\lim_{t\to\infty} P(t) = \lim_{t\to\infty} \frac{MP(0)}{P(0) + (M - P(0))e^{-Mkt}} = \frac{MP(0)}{P(0)} = M,$$

because $\lim_{t\to\infty} e^{-Mkt} = 0$.

Here is the second observation. If the population $P(t)$ starts smaller than the limiting population (i.e. M), then the population will always be smaller than M because the denominator of

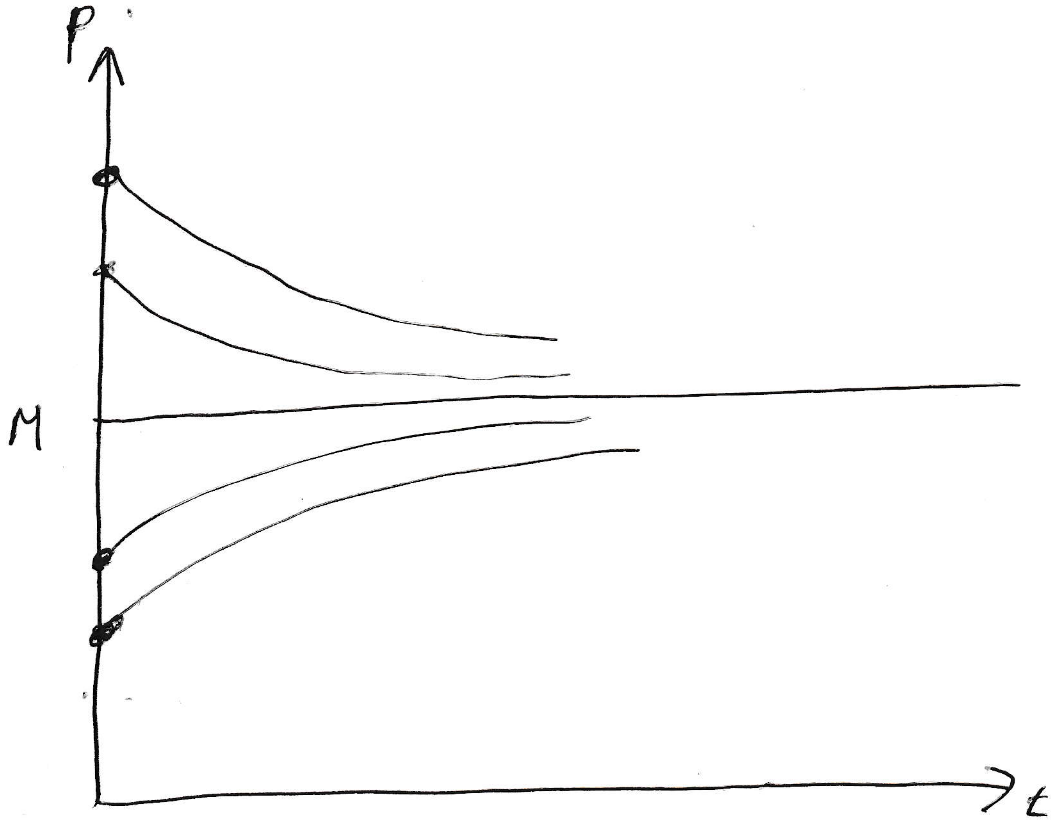
$$P(t) = \frac{MP(0)}{P(0) + (M - P(0))e^{-Mkt}}$$

will always be bigger than $P(0)$; hence the ratio $\frac{P(0)}{P(0)+(M-P(0))e^{-Mkt}}$ will always be LESS THAN 1; so $P(t)$ is some fraction times M.

Similarly, if the population starts larger than the limiting population, then the population will always be larger than M because the multiplier $\frac{P(0)}{P(0)+(M-P(0))e^{-Mkt}}$ will always be larger than one.

The graph of $P(t)$ for various choices of $P(0)$ looks like:

Solutions of the logistic equation



$$\frac{dP}{dt} = kP(M-P)$$ with $0 < k, M$