Problem 18 in Section 2.1 Consider a population $P(t)$ which satisfies the extinction/explosion Differential Equation $\frac{dP}{dt} = aP^2 - bP$, where $B = aP^2$ is the time rate at which births occur and $D = bP$ is the rate at which deaths occur. If the initial population is $P(0) = P_0$ and B_0 births per month and D_0 births per month are occurring at time $t = 0$, show that the threshold population is $M = D_0 P_0 / B_0$.

Solution. Compare the two forms of the extinction-explosion Differential Equation:

$$\frac{dP}{dt} = kP^2 - kMP$$
$$\frac{dP}{dt} = aP^2 - bP$$

to see that $a = k$ and $b = kM$. Plug $t = 0$ into the equations $B = aP^2$ and $D = bP$ to see that $B(0) = aP(0)^2$ and $D(0) = bP(0)$. Conclude that

$$M = \frac{b}{k} = \frac{b}{a} = \frac{\frac{D(0)}{P(0)}}{\frac{B(0)}{P(0)^2}} = \frac{D(0)}{P(0)} \frac{P(0)^2}{B(0)} = \frac{D(0)P(0)}{B(0)} = \frac{D_0 P_0}{B_0}.$$