Problem 16 in Section 2.1 A rabbit population satisfies the Logistic Differential Equation $\frac{dP}{dt} = aP - bP^2$. If the initial population is 120 rabbits and there are 8 births per month and 6 deaths per month occurring at time $t = 0$, how many months does it take for $P(t)$ to reach 95% of the limiting population M?

Solution. We know that the solution of the Initial Value Problem

$$\frac{dP}{dt} = kP(M - P), \quad P(0) = P_0$$

is

$$P(t) = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}}.$$

We first see how k and M are related to a and b. Compare the two Differential Equations:

$$\frac{dP}{dt} = kMP - kP^2$$
$$\frac{dP}{dt} = aP - bP^2$$

to see that $a = kM$ and $b = k$. It follows that

$$M = \frac{a}{k} = \frac{a}{b} = \frac{\frac{B(0)}{P(0)}}{\frac{D(0)}{P(0)^2}} = \frac{B(0)P(0)}{D(0)},$$

because $B(0) = aP(0)$ and $D(0) = bP(0)^2$. In this problem

$$P(0) = 120, \quad B(0) = 8, \quad D(0) = 6,$$

and P_0 is the same as $P(0)$. It follows that

$$M = \frac{B(0)P(0)}{D(0)} = \frac{8(120)}{6} = 160$$

and

$$k = b = \frac{D(0)}{P(0)^2} = \frac{6}{120^2} = \frac{1}{120(20)}$$

Our job is to find the time t when

$$P(t) = \frac{95}{100}M = \frac{95}{100}(160) = \frac{95}{5}(8) = 19(8) = 152.$$

We must find t with

$$152 = \frac{MP_0}{P_0 + (M - P_0)e^{-kMt}},$$

$$152 = \frac{(160)(120)}{120 + (160-120)e^{-\frac{1}{120(20)}(160)t}} = \frac{(160)(120)}{120 + 40e^{-\frac{1}{15}t}} = \frac{(160)(120)}{40(3 + e^{-\frac{1}{15}t})}.$$

We must find t with

$$152 = \frac{(160)(3)}{(3 + e^{-\frac{1}{15}t})}.$$

Multiply both sides by $3 + e^{-1/15t}$ and divide both sides by 152 to obtain

$$3 + e^{-\frac{1}{15}t} = 3(\tfrac{20}{19}).$$

Subtract 3 from each side to obtain

$$e^{-\frac{1}{15}t} = 3(\tfrac{20}{19} - 1)$$

or

$$e^{-\frac{1}{15}t} = 3(\tfrac{1}{19}).$$

Take the logarithm of each side:

$$-\tfrac{1}{15}t = \ln \tfrac{3}{19}.$$

Multiply both sides by -15:

$$t = -15 \ln \tfrac{3}{19} = 15 \ln \tfrac{19}{3}.$$

The population will reach 95% of M after $15 \ln \frac{19}{3}$ months.