Problem 3 in Section 1.6. Solve $xy' = y + 2\sqrt{xy}$.

Solution. This Differential Equation is homogeneous of degree one in x and y. That is, every term has total degree one in x and y. We make a homogeneous substitution. Divide both sides by x^1 to obtain

$$y' = \tfrac{y}{x} + 2\tfrac{\sqrt{xy}}{x}$$

$$y' = \tfrac{y}{x} + 2\sqrt{\tfrac{xy}{x^2}}$$

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Let $v = \frac{y}{x}$. Observe that $xv = y$ and $x\frac{dv}{dx} + v = \frac{dy}{dx}$. Convert the Differential Equation to

$$x\tfrac{dv}{dx} + v = v + 2\sqrt{v}.$$

Subtract v from both sides:

$$x\tfrac{dv}{dx} = 2\sqrt{v}.$$

Divide both sides by $x\sqrt{v}$; multiply both sides by dx; integrate:

$$\int \tfrac{1}{\sqrt{v}}dv = 2\int \tfrac{1}{x}dx$$

$$2\sqrt{v} = 2\ln|x| + C$$

Divide both sides by 2 and replace v with $\frac{y}{x}$:

$$\sqrt{\tfrac{y}{x}} = \ln|x| + \tfrac{C}{2}$$

Square both sides

$$\tfrac{y}{x} = (\ln|x| + \tfrac{C}{2})^2.$$

Let K be the constant $\frac{C}{2}$. Conclude

$$\boxed{y = x(\ln|x| + K)^2.}$$

Check. We compute

$$\begin{aligned} y' &= 2x(\ln|x| + K)\tfrac{1}{x} + (\ln|x| + K)^2 \\ &= 2(\ln|x| + K) + (\ln|x| + K)^2. \end{aligned}$$

We plug y and y' into the Differential Equation $xy' = y + 2\sqrt{xy}$ and see if it works.

The left side becomes $x\Big(2(\ln|x| + K) + (\ln|x| + K)^2\Big)$.

The right side becomes $x(\ln|x| + K)^2 + 2\sqrt{xx(\ln|x| + K)^2}$.

The left side is equal to the right side. Our answer is correct.