Problem 18 in Section 1.6. Solve $(x+y)y' = 1$.

Solution. This Differential Equation has the form $\frac{dy}{dx} = \frac{1}{L}$, where L is the linear expression $x+y$. We make a linear substitution. That is, we let

$$v = x + y.$$

We are promised that after we turn the Differential Equation into a Differential Equation which involves v as a function of x, then we will be able to separate the variables.

If $v = x + y$, then $\frac{dv}{dx} = 1 + \frac{dy}{dx}$. The original Differential Equation has become

$$v(\tfrac{dv}{dx} - 1) = 1.$$

We separate the variables. Divide both sides by v and add 1 to both sides:

$$\tfrac{dv}{dx} = \tfrac{1}{v} + 1.$$

Get a common denominator:

$$\tfrac{dv}{dx} = \tfrac{1+v}{v}.$$

Multiply both sides by $\frac{v}{1+v}$ and multiply both sides by dx:

$$\tfrac{v}{1+v} dv = dx.$$

Use long division (or make the calculation in your head) to see that

$$\frac{v}{1+v} = 1 - \frac{1}{1+v}.$$

Integrate both sides of

$$(1 - \tfrac{1}{1+v})dv = dx$$

to obtain

$$v - \ln|1+v| = x + C.$$

Put $x+y$ in for v to obtain:

$$x + y - \ln|1+x+y| = x + C.$$

We can subtract x from each side to obtain

$$y - \ln|1+x+y| = C.$$

I would like to solve for y, but I am unable to do that. The best I can say is that

Any function $y = y(x)$ which satisfies $y - \ln|1+x+y| = C$ is a solution of the Differential Equation.

Check. We can use implicit differentiation to find $\frac{dy}{dx}$. If $y = y(x)$ is a function which satisfies $y - \ln|1 + x + y| = C$, then one merely takes $\frac{d}{dx}$ of both sides. Whenever one has to take the derivative of y, one writes $\frac{dy}{dx}$:

$$\frac{dy}{dx} - \frac{1+\frac{dy}{dx}}{1+x+y} = 0.$$

Now we solve for $\frac{dy}{dx}$. Multiply both sides by $1 + x + y$:

$$\frac{dy}{dx}(1 + x + y) - (1 + \frac{dy}{dx}) = 0.$$

This expression is linear in $\frac{dy}{dx}$; that is some terms have $\frac{dy}{dx}$; the rest don't. Get all of the terms that involve $\frac{dy}{dx}$ on one side; get all of the rest of the terms on the other side:

$$\frac{dy}{dx}(1 + x + y - 1) = 1.$$

$$\frac{dy}{dx}(x + y) = 1.$$

We have shown that every $y = y(x)$ which satisfies $y - \ln|1 + x + y| = C$ also satisfies the Differential Equation $(x + y)y' = 1$. Our answer is correct.