Problem 9 in Section 1.4. Find the general solution of $(1-x^2)\frac{dy}{dx} = 2y$.

Solution. Separate the variables and integrate:

$$\int \frac{dy}{y} = 2\int \frac{1}{1-x^2}\,dx \tag{1}$$

Of course,

$$\frac{1}{1-x^2} = \frac{-1}{x^2-1} = \frac{-1}{(x-1)(x+1)}.$$

We apply the technique of Partial Fractions.

$$\frac{-1}{(x-1)(x+1)} = \frac{A}{x-1} + \frac{B}{x+1}.$$

Multiply both sides by $(x-1)(x+1)$ to obtain

$$-1 = A(x+1) + B(x-1).$$

This equation holds for all values of x. If we plug in $x=1$, then we learn that $A = \frac{-1}{2}$. If we plug in $x=-1$, then we learn that $B = \frac{1}{2}$. We rewrite (1) as

$$\int \frac{dy}{y} = \int -\frac{1}{x-1} + \frac{1}{x+1}\,dx$$
$$\ln|y| = -\ln|x-1| + \ln|x+1| + C$$

Exponentiate:

$$|y| = e^C \frac{|x+1|}{|x-1|}.$$

In other words,

$$y = \pm e^C \frac{x+1}{x-1}.$$

Of course, $\pm e^C$ is just a constant. Let's call it K.

$$\boxed{y = K\frac{x+1}{x-1}.}$$

Check. We compute

$$(1-x^2)\frac{dy}{dx} = (1-x^2)K\frac{(x-1)-(x+1)}{(x-1)^2} = (1-x)(1+x)K\frac{-2}{(x-1)^2}$$
$$= 2K\frac{1+x}{x-1} = 2y.\checkmark$$