Problem 35 in Section 1.4. Carbon extracted from an ancient skull contained only one-sixth as much ^{14}C as carbon extracted from present-day bone. How old is the skull?

Solution. The first piece of background is the fact that there are two kinds of Carbon: ^{12}C and ^{14}C: ^{14}C is radioactive and ^{12}C is stable. It is assumed that living things have a constant ratio of the two isotopes because breathing replenishes the ^{14}C which has decayed. But once an organism dies, then ^{14}C continues to decay but is not replenished any longer.

The second piece of background is the assumption that radioactive material decays at a rate proportional to the amount present. (That is, if $A(t)$ is the amount of ^{14}C present at time t, then $\frac{dA}{dt} = -kA$ for some positive constant k.)

The third piece of background is that the k is never given explicitly, but the half-life of each radioactive substance can be looked up. One can calculate the k from the half-life. The half-life of ^{14}C is about 5700 years. (Of course, you do not have to know this.)

Once you have all of the background, the problem is not very hard. Let $A(t)$ be the amount of ^{14}C in the skull at time t. Take $t = 0$ to be the moment the creature died. Measure time in years. Let t_{now} represent now. Let $A(0) = A_0$. We are told that $A(5700) = \frac{1}{2}A_0$ and $A(t_{\text{now}}) = \frac{1}{6}A_0$.

To solve the Differential Equation $\frac{dA}{dt} = -kA$, separate the variables and integrate:

$$\int \frac{dA}{A} = -k \int dt$$

$$\ln A = -kt + C$$

(We know that A is positive so $|A| = A$.) Exponentiate

$$A = e^C e^{-kt}$$

Plug in $t = 0$ to see that

$$A_0 = A(0) = e^C;$$

hence,

$$A(t) = A_0 e^{-kt}.$$

We can use the half life to find k:

$$\tfrac{1}{2}A_0 = A(5700) = A_0 e^{-5700k}$$

$$\tfrac{1}{2} = e^{-5700k}$$

$$\frac{\ln(\frac{1}{2})}{-5700} = k.$$

Now, we find t_{now}.

$$\tfrac{1}{6}A_0 = A(t_{\text{now}}) = A_0 e^{-kt_{\text{now}}}$$

$$\tfrac{1}{6} = e^{-kt_{\text{now}}}$$

$$\ln(\tfrac{1}{6}) = -kt_{\text{now}}$$

$$t_{\text{now}} = \frac{\ln(\frac{1}{6})}{-k} = 5700\frac{\ln\frac{1}{6}}{\ln\frac{1}{2}} = 5700\frac{(-\ln 6)}{(-\ln 2)} = \boxed{5700\frac{\ln 6}{\ln 2}}.$$

It has been about 14,734 years since that animal died.