Problem 21 in Section 1.4. Solve the Initial Value Problem

$$2y\frac{dy}{dx} = \frac{x}{\sqrt{x^2-16}} \quad \text{and} \quad y(5) = 2.$$

Solution. Separate the variables and integrate:

$$\int 2y\,dy = \int \frac{x}{\sqrt{x^2-16}}\,dx$$

Substitute. Let $u = x^2 - 16$. It follows that $du = 2x\,dx$

$$y^2 = \sqrt{x^2-16} + C$$

$$y = \pm\sqrt{\sqrt{x^2-16} + C}$$

Plug in the initial condition

$$2 = \pm\sqrt{\sqrt{25-16} + C}.$$

We see that $\pm$ must be $+$ and

$$2 = \sqrt{3 + C}.$$

Thus $C = +1$ and

$$\boxed{y = \sqrt{\sqrt{x^2-16} + 1}}$$

Check. We compute

$$\frac{dy}{dx} = \left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{\sqrt{x^2-16}+1}}\right)\left(\frac{1}{2}\right)\frac{2x}{\sqrt{x^2-16}}.$$

Thus,

$$2y\frac{dy}{dx} = 2\left(\sqrt{\sqrt{x^2-16}+1}\right)\left(\frac{1}{2}\right)\left(\frac{1}{\sqrt{\sqrt{x^2-16}+1}}\right)\left(\frac{1}{2}\right)\frac{2x}{\sqrt{x^2-16}}$$

$$= \frac{x}{\sqrt{x^2-16}} \checkmark$$

and

$$y(5) = \sqrt{\sqrt{5^2-16}+1} = \sqrt{3+1} = 2, \checkmark$$