Problem 13 in Section 1.4. Find the general solution of $y^3 \frac{dy}{dx} = (y^4+1)\cos x$.

Solution. Separate the variables and integrate:

$$\int \frac{y^3}{y^4+1} = \int \cos x \, dx.$$

On the left side one substitutes. Let $u = y^4 + 1$. It follows that $du = 4dy$.

$$\frac{1}{4}\ln(y^4+1) = \sin x + C$$

(The expression $y^4 + 1$ is positive, so $|y^4+1| = y^4+1$.)

$$\ln(y^4+1) = 4\sin x + 4C$$

Exponentiate:

$$y^4 + 1 = e^{4C} e^{4\sin x}$$

Let K be the constant e^{4C}.

$$\boxed{y = \pm(Ke^{4\sin x} - 1)^{1/4}}.$$

Check. Fix ϵ equal to either $+1$ or -1. Let $y = \epsilon(Ke^{4\sin x} - 1)^{1/4}$. We calculate

$$y^3 \frac{dy}{dx} = \epsilon^3 (Ke^{4\sin x} - 1)^{3/4} \epsilon (\frac{1}{4})(Ke^{4\sin x} - 1)^{-3/4} 4(\cos x) K e^{4\sin x}$$

The 4's cancel. The terms $(Ke^{4\sin x} - 1)^{-3/4}$ and $(Ke^{4\sin x} - 1)^{3/4}$ multiply to 1. The number ϵ^4 is equal to 1. So,

$$y^3 \frac{dy}{dx} = K(\cos x)e^{4\sin x}.$$

On the other hand,

$$(y^4+1)\cos x = (Ke^{4\sin x} - 1 + 1)\cos x.$$

These are equal. Our proposed answer is correct.