Problem 46 in Section 1.1. Suppose the velocity v of a motorboat in water satisfies the Differential Equation $\frac{dv}{dt} = kv^2$. The initial speed of the boat is $v(0) = 10$ meters per second (m/s) and v is decreasing at the rate of 1 m/s^2 when $v = 5$m/s. How long does it take for the velocity of the boat to decrease to 1m/s? To 1/10 m/s? When does the boat stop?

Solution. Measure distance in meters and time in seconds. We are given the Differential Equation

$$\frac{dv}{dt} = kv^2$$

and the initial condition $v(0) = 10$. We think of this as an Initial Value Problem. We can solve the IVP and express v as a function of t. Of course this function will involve the constant k. We use the information $\frac{dv}{dt}|_{v=5} = -1$. to evaluate k. Once we have the formula for $v(t)$, then it is not difficult to answer the questions.

We find k. We know that $\frac{dv}{dt} = kv^2$ always and we know that when $v = 5$, then $\frac{dv}{dt} = -1$. It follows $-1 = k(25)$; so $k = \frac{-1}{25}$.

We solve the Differential Equation by separating the variables and integrating:

$$\int \frac{dv}{v^2} = \int k\,dt$$

$$-\frac{1}{v} = kt + C$$

We use the initial condition to evaluate C:

$$-\frac{1}{10} = C$$

So,

$$v = -\frac{1}{kt + C} = -\frac{1}{\frac{-1}{25}t - \frac{1}{10}} = \frac{250}{10t - 25}$$

Now we can answer the questions.

The velocity is 1m/s when

$$\frac{250}{10t + 25} = 1$$

$$250 = 10t + 25$$

$$\boxed{t = \frac{225}{10}\text{ sec}}$$

The velocity is $\frac{1}{10}$m/s when

$$\frac{250}{10t + 25} = \frac{1}{10}$$

$$\boxed{t = \frac{2475}{10}\text{ sec}}$$ (This time seems unreasonable!)

The velocity is zero when

$$\frac{250}{10t+25} = 0$$

Oops, that never happens. This boat will never stop!

The real point of the problem is that $\frac{dv}{dt} = kv^2$ (the deceleration due to air and water resistance is proportional to the **square** of the velocity) might work very well when v is large (because when the velocity is large then v^2 is REALLY large, and this makes sense because there is an awful lot of resistance). On the other hand, when v is near zero, then v^2 is really really near zero. One has to use a different Differential Equation to model the resistance force when v is near zero.