Math 241, Final Exam, Summer, 2002

1. Let $f(x,y)=x\sin(xy)$. Find $\overrightarrow{\boldsymbol{\nabla}} f$.

We see that

$$\overrightarrow{\boldsymbol{\nabla}} f = f_x \overrightarrow{\boldsymbol{i}} + f_y \overrightarrow{\boldsymbol{j}} = \boxed{\Big(xy\cos(xy)+\sin(xy)\Big)\,\overrightarrow{\boldsymbol{i}} + x^2\cos(xy)\,\overrightarrow{\boldsymbol{j}}.}$$

2. Find the equations of the line through the points $P=(1,-3,4)$ and $Q=(3,4,6)$. **Check your answer.**

We calculate $\overrightarrow{PQ} = 2\overrightarrow{\boldsymbol{i}} + 7\overrightarrow{\boldsymbol{j}} + 2\overrightarrow{\boldsymbol{k}}$ **; so the line is**

$$\boxed{\begin{cases} x = 1+2t \\ y = -3+7t \\ z = 4+2t \end{cases}}$$

Notice that when $t=0$ **the point is** $(1,-3,4)$ **and when** $t=1$**, the point is** $(3,4,6)$ **.**

3. Find the equation of the plane through the points $P=(2,1,2)$, $Q=(3,3,6)$, and $R=(0,-1,0)$. **Check your answer.**

We calculate $\overrightarrow{PQ} = \overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}} + 4\overrightarrow{\boldsymbol{k}}$ **and** $\overrightarrow{PR} = -2\overrightarrow{\boldsymbol{i}} - 2\overrightarrow{\boldsymbol{j}} - 2\overrightarrow{\boldsymbol{k}}$ **; so**

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ 1 & 2 & 4 \\ -2 & -2 & -2 \end{vmatrix} = \begin{vmatrix} 2 & 4 \\ -2 & -2 \end{vmatrix}\overrightarrow{\boldsymbol{i}} - \begin{vmatrix} 1 & 4 \\ -2 & -2 \end{vmatrix}\overrightarrow{\boldsymbol{j}} + \begin{vmatrix} 1 & 2 \\ -2 & -2 \end{vmatrix}\overrightarrow{\boldsymbol{k}}$$

$= 4\overrightarrow{\boldsymbol{i}} - 6\overrightarrow{\boldsymbol{j}} + 2\overrightarrow{\boldsymbol{k}}$ **. The plane is** $4(x-0)-6(y+1)+2(z-0)=0$ **, which is the same as**

$$\boxed{2x-3y+z=3.}$$

Plug in P**:** $4-3+2=3$ **. Plug in** Q**:** $6-9+6=3$ **. Plug in** R**:** $3=3$ **.**

4. Let $f(x,y)=\dfrac{x^2}{x^2+2y^2}$. Calculate the limit of $f(x,y)$ as $(x,y)\to(0,0)$ along $y=3x$.

We calculate

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along}\quad y=3x}} \frac{x^2}{x^2+2y^2} = \lim_{x\to 0}\frac{x^2}{x^2+2(3x)^2} = \lim_{x\to 0}\frac{x^2}{19x^2} = \lim_{x\to 0}\frac{1}{19} = \boxed{\frac{1}{19}.}$$

5. Identify all local extreme points and all saddle points of
$f(x,y) = x^2y - 6y^2 - 3x^2$.

We claculate $f_x = 2xy - 6x = 2x(y-3)$ **and** $f_y = x^2 - 12y$ **. Both partial derivatives are zero when**

$$\begin{cases} 0 = 2x(y-3) \\ 0 = x^2 - 12y \end{cases}$$

So either $x = 0$ **and** $0 = -12y$ **; or** $y = 3$ **and** $0 = x^2 - 36$ **. We conclude that there are three points where** f_x **and** f_y **both are zero; namely,** $(0,0)$ **,** $(6,3)$ **, and** $(-6,3)$ **.**

We now calculate $f_{xx} = 2y - 6$ **,** $f_{xy} = 2x$ **, and** $f_{yy} = -12$ **. The discriminant** D **is equal to**

$$D = f_{xx}f_{yy} - f_{xy}^2 = (2y-6)(-12) - (2x)^2.$$

At $(0,0)$ **,** $D(0,0) = 72 > 0$ **and** $f_{yy}(0,0) = -12 < 0$ **; so** $(0,0,0)$ **is a local maximum point. Observe also that** $D(-6,3) = D(6,3) = -144 < 0$ **; thus,** $(6,3,f(6,3))$ **and** $(-6,3,f(-6,3))$ **both are saddle points. We conclude**

$(0,0,0)$ **is a local maximum point, and**
$(6,3,f(6,3))$ **and** $(-6,3,f(-6,3))$ **both are saddle points.**

6. Find the intersection of the two lines:

$$\frac{x-5}{2} = \frac{y-3}{1} = \frac{z}{-1} \quad \text{and} \quad \frac{x+8}{3} = \frac{y+5}{2} = \frac{z+1}{1}.$$

Check your answer.

Walk on the first line. At time t **, you stand at the point** $x = 5 + 2t$ **,** $y = 3 + t$ **, and** $z = -t$ **. We look for a time which causes your position to satisfy both equations of the second line. We need a common solution to**

$$\frac{5+2t+8}{3} = \frac{3+t+5}{2} = \frac{-t+1}{1}.$$

So, we need

$$\frac{13+2t}{3} = \frac{t+8}{2} \quad \textbf{and} \quad \frac{13+2t}{3} = \frac{-t+1}{1}.$$

We need

$$26 + 4t = 3t + 24 \quad \textbf{and} \quad 13 + 2t = -3t + 3.$$

Both equations hold for $t = -2$ **. Our position at** $t = -2$ **is**

$$\boxed{(1,1,2).}$$

We check that this point satisfies the equations of the first line:

$$\frac{1-5}{2} = \frac{1-3}{1} = \frac{2}{-1}.$$

We check that this point satisfies the equations of the second line:

$$\frac{1+8}{3} = \frac{1+5}{2} = \frac{2+1}{1}.$$

9. Compute the directional derivative $D_{\vec{u}} f$ at the point $(3,2)$ in the direction of the unit vector $\vec{\boldsymbol{u}} = \frac{5}{13}\vec{\boldsymbol{i}} + \frac{12}{13}\vec{\boldsymbol{j}}$ for $f(x,y) = 3x^2y^4$.

$$D_{\vec{u}} f(3,2) = \vec{\nabla} f(3,2) \cdot \vec{\boldsymbol{u}} = (6xy^4\,\vec{\boldsymbol{i}} + 12x^2y^3)|_{(3,2)} \cdot \vec{\boldsymbol{u}}$$
$$= \left((18)(16)\,\vec{\boldsymbol{i}} + (108)(8)\,\vec{\boldsymbol{j}}\right) \cdot \left(\frac{5}{13}\vec{\boldsymbol{i}} + \frac{12}{13}\vec{\boldsymbol{j}}\right) = \frac{(18)(16)(5) + (108)(8)(12)}{13}$$

10. Where does the line normal to $x^2 + 2y^2 + 3z^2 = 9$ at $(2,1,-1)$ intersect $2x + y - z + 3 = 0$?

Gradients are perpendicular to level sets. The ellipsoid is level 9 of the function which sends (x,y,z) to $x^2 + 2y^2 + 3z^2$. The gradient of the left side is $2x\,\vec{\boldsymbol{i}} + 4y\,\vec{\boldsymbol{j}} + 6z\,\vec{\boldsymbol{k}}$. The gradient evaluated at $(2,1,-1)$ is $4\,\vec{\boldsymbol{i}} + 4\,\vec{\boldsymbol{j}} - 6\,\vec{\boldsymbol{k}}$. The normal line is

$$\begin{cases} x = 2 + 4t \\ y = 1 + 4t \\ z = -1 - 6t \end{cases}$$

The line hits the plane when

$$2(2+4t) + (1+4t) - (-1-6t) + 3 = 0;$$

that is, $t = -1/2$. The normal line hits the plane at the point $\boxed{(0,-1,2)}$. Check that $(0,-1,2)$ is on the plane: $0 - 1 - 2 + 3 = 0$. The vector from $(2,1,-1)$ to $(0,-1,2)$ is $<-2,-2,3>$, which is parallel to $4\,\vec{\boldsymbol{i}} + 4\,\vec{\boldsymbol{j}} - 6\,\vec{\boldsymbol{k}}$.

11. Compute $\iint_R (x^2+2y)\,dA$, where R is the region between $y = x^2$ and $y = \sqrt{x}$.

A picture maybe found on a separate page. The integral is equal to

$$\int_0^1\int_{x^2}^{\sqrt{x}}(x^2+2y)dy\,dx=\int_0^1(x^2y+y^2)\Big|_{x^2}^{\sqrt{x}}dx=\int_0^1(x^{5/2}+x-2x^4)dx=$$

$$\left(\frac{2}{7}x^{7/2}+\frac{x}{2}-\frac{2}{5}x^5\right)\Big|_0^1=\boxed{\frac{2}{7}+\frac{1}{2}-\frac{2}{5}}$$

12. Find the volume of the solid which is between $z=16-x^2-y^2$ and the xy-plane.

The base is the circle $x^2+y^2=16$ in the xy-plane. The top is $z=16-x^2-y^2$. I will do the integral in polar coordinates. The volume is the integral over the base of the top, which is equal to

$$\int_0^{2\pi}\int_0^4 r(16-r^2)dr\,d\theta=\int_0^{2\pi}\int_0^4(16r-r^3)dr\,d\theta=\int_0^{2\pi}(8r^2-r^4/4)\Big|_0^4 d\theta$$

$$=\int_0^{2\pi}(8r^2-r^4/4)\Big|_0^4 d\theta=\quad(8(16)-16(4))\theta\Big|_0^{2\pi}=16(4)2\pi=\boxed{128\pi}.$$

13. Compute $\iint_R x^2dA$, where R is the region between $x^2+y^2=1$ and $x^2+y^2=4$.

I will do this integral in polar coordinates. The integral is equal to

$$\int_0^{2\pi}\int_1^2 r^3\cos^2\theta\,dr\,d\theta=\int_0^{2\pi}\frac{r^4}{4}\,\frac{1+\cos 2\theta}{2}\Big|_1^2 d\theta=\quad\frac{15}{8}\left(\theta+\frac{\sin 2\theta}{2}\right)\Big|_0^{2\pi}$$

$$=\frac{15}{8}(2\pi)=\boxed{\frac{15\pi}{4}}.$$

14. Compute $\int_0^2\int_0^{\sqrt{4-y^2}}x^2dx\,dy$.

This integral is over the quarter circle in the first quadrant of the circle of radius 2, with center at the origin. I will do the integral in polar coordinates. The integral is equal to

$$\int_0^{\pi/2}\int_0^2 r^3\cos^2\theta dr\,d\theta=\int_0^{\pi/2}\frac{r^4}{4}\,\frac{1+\cos 2\theta}{2}\Big|_0^2 d\theta=\left(2\theta+\frac{\sin 2\theta}{2}\right)\Big|_0^{\pi/2}=\boxed{\pi}.$$

15. Compute $\int_C (x+y+z)\,dx + x\,dy - yz\,dz$, where C is the line segment from $(1,2,1)$ to $(2,1,0)$.

I will parameterize the line segment by letting (x,y,z) be the point with the property that the vector $\overrightarrow{(1,2,1)(x,y,z)}$ is equal to t times the vector $\overrightarrow{(1,2,1)(2,1,0)}$ That is, $<x-1, y-2, z-1> = t<1,-1,-1>$. In other words, $x = t+1$, $y = -t+2$, and $z = -t+1$ for $0 \le t \le 1$. The integral is equal to

$$\int_0^1 \Big((t+1-t+2-t+1) + (t+1)(-1) - (-t+2)(-t+1)(-1)\Big)\,dt$$

$$= \int_0^1 (4-t-t-1+t^2-3t+2)\,dt = \int_0^1 (t^2-5t+5)\,dt = \left(\frac{t^3}{3} - \frac{5t^2}{2} + 5t\right)\Bigg|_0^1$$

$$= \boxed{\frac{1}{3} - \frac{5}{2} + 5.}$$

16. Let $\vec{a} = 1\,\vec{i} + 2\,\vec{j} + 3\,\vec{k}$ and $\vec{b} = 4\,\vec{i} + 4\,\vec{j} + 10\,\vec{k}$. Find vectors $\vec{u}$ and $\vec{v}$ with $\vec{b} = \vec{u} + \vec{v}$, $\vec{u}$ parallel to $\vec{a}$, and $\vec{v}$ perpendicular to $\vec{a}$. (Every number in the answer is an integer. If you have fractions, either you can rid of them or you have made a mistake.) **Check your answer**

The vector $\vec{u}$ is equal to

$$\text{proj}_{\vec{a}}\,\vec{b} = \frac{\vec{a}\cdot\vec{b}}{\vec{a}\cdot\vec{a}}\,\vec{a} = \frac{4+8+30}{1+4+9}\,\vec{a} = \frac{42}{14}\,\vec{a} = 3(1\,\vec{i} + 2\,\vec{j} + 3\,\vec{k})$$
$$= \boxed{3\,\vec{i} + 6\,\vec{j} + 9\,\vec{k} = \vec{u}}$$

and

$$\vec{v} = \vec{b} - \vec{u} = 4\,\vec{i} + 4\,\vec{j} + 10\,\vec{k} - (3\,\vec{i} + 6\,\vec{j} + 9\,\vec{k}) = \boxed{\vec{i} - 2\,\vec{j} + \vec{k} = \vec{v}}.$$

Check that $\vec{v}\cdot\vec{a} = 0$: $1-4+3=0$.

17. Graph and name $x^2+y^2-z^2=1$ in 3-space.

This is a hyperboloid of one sheet. The picture appears on a separate page.

18. Graph and describe the graph of $yz=0$ in 3-space.

The graph is the union of the xz-plane together with the xy-plane. The picture appears on a separate page.

19. Find the equation of the line tangent to the curve parameterized by $\overrightarrow{\boldsymbol{r}}(t) = 3t^2\,\overrightarrow{\boldsymbol{i}} + t^3\,\overrightarrow{\boldsymbol{j}}$ at $t = 2$

At $t = 2$, the position is $(12, 8)$. We see that $\overrightarrow{\boldsymbol{r}}'(t) = 6t\,\overrightarrow{\boldsymbol{i}} + 3t^2\,\overrightarrow{\boldsymbol{j}}$; so, $\overrightarrow{\boldsymbol{r}}'(2) = 12\,\overrightarrow{\boldsymbol{i}} + 12\,\overrightarrow{\boldsymbol{j}}$. The tangent line is

$$\boxed{x = 12 + 12t,\ y = 8 + 12t,\ z = 0}.$$

20. Find the equation of the plane tangent to $z = x^2 + y^2$ at the point where $x = 3$ and $y = 4$.

When $x = 3$ and $y = 4$, the z-coordinate of the point is 25. Gradients are perpendicular to level sets. The surface is level zero of the function $(x, y, z) \mapsto x^2 + y^2 - z$. The gradient of the defining function is $2x\,\overrightarrow{\boldsymbol{i}} + 2y\,\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}$. The gradient of the defining function, evaluated at $(3, 4)$ is $6\,\overrightarrow{\boldsymbol{i}} + 8\,\overrightarrow{\boldsymbol{j}} - vk$. The tangent plane is

$$\boxed{6(x - 3) + 8(y - 4) - (z - 25) = 0}.$$