Math 241 Exam 3 Spring 2008 Solutions

Please leave room in the upper left corner for the staple.
TAKE THESE QUESTIONS HOME WITH YOU WHEN YOU LEAVE.
I WILL POST SOLUTIONS LATER TODAY.

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 7 problems. Most of the problems are worth 7 points. The exam is worth 50 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

1. **Find the directional derivative of** $f(x,y) = xe^{xy}$ **at the point** $(2,3)$ **in the direction of the vector** $\overrightarrow{\boldsymbol{a}} = 3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}}$ **.**

Let $\overrightarrow{\boldsymbol{u}}$ be the unit vector in the direction of $\overrightarrow{\boldsymbol{a}}$. So, $\overrightarrow{\boldsymbol{u}} = \overrightarrow{\boldsymbol{a}}/||\overrightarrow{\boldsymbol{a}}|| = \overrightarrow{\boldsymbol{a}}/5$. We have

$$D_{\overrightarrow{\boldsymbol{u}}} f(2,3) = \overrightarrow{\nabla} f(2,3) \cdot \overrightarrow{\boldsymbol{u}} = ((xye^{xy} + e^{xy})\overrightarrow{\boldsymbol{i}} + x^2e^{xy}\overrightarrow{\boldsymbol{j}})|_{(2,3)} \cdot \overrightarrow{\boldsymbol{u}}$$

$$= ((6e^6 + e^6)\overrightarrow{\boldsymbol{i}} + 4e^6\overrightarrow{\boldsymbol{j}}) \cdot (3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}})/5 = e^6(21+16)/5 = \boxed{37e^6/5.}$$

2. **Find the equation of the plane tangent to** $z = x^2 + y^2$ **when** $x = 1$ **and** $y = 2$ **.**

Gradients are perpendicular to level sets. So write the surface as a level set: $0 = x^2 + y^2 - z$. Let $f(x,y,z)$ be the function $f(x,y,z) = x^2 + y^2 - z$. Our surface is the level set $f = 0$. We have $\overrightarrow{\nabla} f(1,2,5)$ is perpendicular to the tangent plane. We compute

$$\overrightarrow{\nabla} f(1,2,5) = (2x\overrightarrow{\boldsymbol{i}} + 2y\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}})(1,2,5) = 2\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}.$$

The tangent plane passes through $(1,2,5)$ and is perpendicular to $2\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}$. The tangent plane is

$$\boxed{2(x-1) + 4(y-2) - (z-5) = 0.}$$

3. **Find all relative maxima, relative minima, and saddle points of** $f(x,y) = y^2 + xy + 3y + 2x + 3$ **.**

We compute

$$f_x = y + 2, \quad f_y = 2y + x + 3, \quad f_{xx} = 0, \quad f_{xy} = 1, \quad f_{yy} = 2.$$

We see that both partial derivatives f_x and f_y are zero when $y + 2 = 0$ and $2y + x + 3 = 0$. This happens when $y = -2$ and $x = 1$. We see that $f_{xx}f_{yy} - f_{xy}^2$ is always -1. We conclude that $\boxed{(1,-2,3) \text{ is a saddle point of } z = f(x,y)}$.

4. **(8 points) Find the points on the sphere** $x^2 + y^2 + z^2 = 36$ **that are closest to and farthest from the point** $(1,2,2)$ **.**

We must find the extreme points of the function

$$f(x,y,z) = (x-1)^2 + (y-2)^2 + (z-2)^2$$

subject to the constraint $g(x,y,z) = 36$, where $g(x,y,z) = x^2 + y^2 + z^2$. The constraint is bounded, the constraint has no boundary, and $\overrightarrow{\nabla} g$ is never zero on the constraint. Thus, the maximum and the minimum of f on the constraint occur at points where $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$ for some number λ. We find all points on $g = 36$ where

$$2(x-1)\,\overrightarrow{\boldsymbol{i}} + 2(y-2)\,\overrightarrow{\boldsymbol{j}} + 2(z-2)\,\overrightarrow{\boldsymbol{k}} = \lambda(2x\,\overrightarrow{\boldsymbol{i}} + 2y\,\overrightarrow{\boldsymbol{j}} + 2z\,\overrightarrow{\boldsymbol{k}}).$$

So, we solve

$$\begin{cases} 2(x-1) = 2x\lambda \\ 2(y-2) = 2y\lambda \\ 2(z-2) = 2z\lambda \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

simultaneously. Notice that λ can not equal 1 because $2 \neq 0$. We solve

$$\begin{cases} (1-\lambda)x = 1 \\ (1-\lambda)y = 2 \\ (1-\lambda)z = 2 \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

simultaneously. We solve

$$\begin{cases} x = 1/(1-\lambda) \\ y = 2/(1-\lambda) \\ z = 2/(1-\lambda) \\ x^2 + y^2 + z^2 = 36 \end{cases}$$

simultaneously. We solve

$$\begin{cases} x = 1/(1-\lambda) \\ y = 2/(1-\lambda) \\ z = 2/(1-\lambda) \\ \frac{1}{(1-\lambda)^2}(1+4+4) = 36 \end{cases}$$

simultaneously. We solve

$$\begin{cases} x = 1/(1-\lambda) \\ y = 2/(1-\lambda) \\ z = 2/(1-\lambda) \\ \frac{1}{(1-\lambda)^2} = 4 \end{cases}$$

simultaneously. We solve

$$\begin{cases} x = 1/(1-\lambda) \\ y = 2/(1-\lambda) \\ z = 2/(1-\lambda) \\ \frac{1}{4} = (1-\lambda)^2 \end{cases}$$

simultaneously. We solve

$$\begin{cases} x = 1/(1-\lambda) \\ y = 2/(1-\lambda) \\ z = 2/(1-\lambda) \\ \pm\frac{1}{2} = (1-\lambda) \end{cases}$$

simultaneously. So, λ is equal to $1/2$ or $3/2$. If $\lambda = 1/2$, then the point is $(2,4,4)$. If $\lambda = 3/2$, then the point is $(-2,-4,-4)$. We see that $f(2,4,4) = 9$ and $f(-2,-4,-4) = 81$. So

$$\boxed{\begin{array}{c} (2,4,4) \text{ is the point on } x^2+y^2+z^2=36 \text{ closest to } (1,2,2) \\ (-2,-4,-4) \text{ is the point on } x^2+y^2+z^2=36 \text{ farthest from } (1,2,2) \end{array}}.$$

5. **Find $\iint_R \sin(y^3)dA$, where R is the region in the xy-plane bounded by $y = \sqrt{x}$, $y = 2$, and $x = 0$.**

Look at the picture. The integral is equal to

$$\int_0^2 \int_0^{y^2} \sin(y^3)\,dx\,dy = \int_0^2 y^2 \sin(y^3)dy = -(\cos(y^3))/3\Big|_0^2 = \boxed{(1/3)(1-\cos 8).}$$

6. **Find the volume of the region between** $z = 9 - x^2 - y^2$ **and** $z = 0$ **.**

Look at the picture. The volume is equal to

$$\int_0^{2\pi} \int_0^3 r(9 - r^2)\, dr\, d\theta = \int_0^{2\pi} (9r^2/2 - r^4/4)\Big|_0^3 \, d\theta = 2\pi(81/2 - 81/4) = 2\pi(81/4)$$

$$= \boxed{81\pi/2.}$$

7. **Find** $\iiint_G (1 - x^2 - y^2 - z^2) dV$ **, where** G **is the region inside the sphere** $x^2 + y^2 + z^2 = 1$ **.**

I use spherical coordinates. The region G is all points (ρ, θ, ϕ) with $0 \le \rho \le 1$, $0 \le \theta \le 2\pi$ and $0 \le \phi \le \pi$. In sperical coordinates, $\rho = x^2 + y^2 + z^2$, and the Jacobian is equal to $\rho^2 \sin\phi$ so $dx\, dy\, dz$ becomes $\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta$. The integral is equal to

$$\int_0^{2\pi} \int_0^{\pi} \int_0^1 (1 - \rho^2)\rho^2 \sin\phi\, d\rho\, d\phi\, d\theta = 2\pi \int_0^{\pi} (\rho^3/3 - \rho^5/5)\Big|_0^1 \sin\phi\, d\phi$$

$$= 2\pi(2/15)(-\cos\phi)\Big|_0^{\pi} = 2\pi(2/15)(-(-1) - (-1)) = \boxed{8\pi/15.}$$