Math 241 Exam 2 Spring 2008 Solution

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 7 problems. Most of the problems are worth 7 points. The exam is worth 50 points. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

1. **Find the directional derivative of** $f(x,y) = x\ln(x+y)$ **at the point** $(1,2)$ **in the direction of the vector** $\overrightarrow{\boldsymbol{a}} = 2\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}}$ **.**

Let $\overrightarrow{\boldsymbol{u}}$ be the unit vector in the direction of $\overrightarrow{\boldsymbol{a}}$. So, $\overrightarrow{\boldsymbol{u}} = \overrightarrow{\boldsymbol{a}}/||\overrightarrow{\boldsymbol{a}}|| = \overrightarrow{\boldsymbol{a}}/\sqrt{13}$. We have

$$D_{\overrightarrow{\boldsymbol{u}}} f(1,2) = \overrightarrow{\nabla} f(1,2)\cdot\overrightarrow{\boldsymbol{u}} = \left[\left(\frac{x}{x+y} + \ln(x+y)\right)\overrightarrow{\boldsymbol{i}} + \frac{x}{x+y}\overrightarrow{\boldsymbol{j}}\right](1,2)\cdot\overrightarrow{\boldsymbol{u}}$$

$$= \left[(1/3+\ln 3)\overrightarrow{\boldsymbol{i}} + 1/3\overrightarrow{\boldsymbol{j}}\right]\cdot(2\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}})/\sqrt{13} = \boxed{[2(1/3+\ln 3)+1]/\sqrt{13}.}$$

2. **Find the equation of the plane tangent to** $z^2 = x^2 + y^2$ **at the point** $(3,4,5)$ **.**

Gradients are perpendicular to level sets. So write the surface as a level set: $0 = x^2 + y^2 - z^2$. Let $f(x,y,z)$ be the function $f(x,y,z) = x^2 + y^2 - z^2$. Our surface is the level set $f = 0$. We have $\overrightarrow{\nabla} f(3,4,5)$ is perpendicular to the tangent plane. We compute

$$\overrightarrow{\nabla} f(3,4,5) = (2x\overrightarrow{\boldsymbol{i}} + 2y\overrightarrow{\boldsymbol{j}} - 2z\overrightarrow{\boldsymbol{k}})(3,4,5) = 6\overrightarrow{\boldsymbol{i}} + 8\overrightarrow{\boldsymbol{j}} - 10\overrightarrow{\boldsymbol{k}}.$$

The tangent plane passes through $(3,4,5)$ and is perpendicular to $6\overrightarrow{\boldsymbol{i}} + 8\overrightarrow{\boldsymbol{j}} - 10\overrightarrow{\boldsymbol{k}}$. The tangent plane is

$$\boxed{6(x-3) + 8(y-4) - 10(z-5) = 0.}$$

3. **Find all points of intersection of the line** $x = -1+t$ **,** $y = 2+t$ **,** $z = 2t+7$ **and the surface** $z = x^2 + y^2$ **.**

I think of a creature walking along the line. The position of the creature is given by: at time t the x-coordinate of the creature is $x = -1+t$; the y-coordinate of the creature is $y = 2+t$ and the z-coordinate is $z = 2t+7$. We first find out WHEN the creature is on the surface. In other words, we solve

$$2t + 7 = (-1+t)^2 + (2+t)^2$$

$$2t + 7 = 1 - 2t + t^2 + 4 + 4t + t^2$$

$$0 = 2t^2 - 2 = 2(t^2 - 1) = 2(t-1)(t+1).$$

The creature stands on the surface at time $t = -1$ and also at time $t = 1$. At $t = 1$ the creature stands on $\boxed{(0,3,9)}$, at time $t = -1$, the creature stands on $\boxed{(-2,1,5)}$. Both points are on the surface because $9 = 0 + 9$ and $5 = 4 + 1$.

4. **Find the equation of the plane that contains the lines** $x = -2+t$ **,** $y = 3+2t$ **,** $z = 4-t$ **, and** $x = 3-t$ **,** $y = 4-2t$ **,** $z = t$ **.**

The two lines are not equal because $P = (-2,3,4)$ is on the first line but not the second line. The vector $\overrightarrow{\boldsymbol{v}} = \overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}$ is parallel to the first line. The vector $\overrightarrow{\boldsymbol{w}} = -\overrightarrow{\boldsymbol{i}} - 2\overrightarrow{\boldsymbol{j}} + \overrightarrow{\boldsymbol{k}}$ is parallel to the second line. The vector $\overrightarrow{\boldsymbol{w}}$ is a multiple of $\overrightarrow{\boldsymbol{v}}$. Thus the two lines are parallel and there really is exactly one plane which contains them.

Observe that the point $Q = (3,4,0)$ is on the second line. We see that $\overrightarrow{PQ}$ and $\overrightarrow{\boldsymbol{v}}$ are both parallel to the plane. So

$$\overrightarrow{PQ} \times \overrightarrow{\boldsymbol{v}} = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ 5 & 1 & -4 \\ 1 & 2 & -1 \end{vmatrix} = 7\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} + 9\overrightarrow{\boldsymbol{k}}$$

is parallel to the plane. The plane is

$$7(x-3) + (y-4) + 9z = 0,$$

which is the same as

$$\boxed{7x + y + 9z = 25.}$$

We check

$$7(-2+t) + (3+2t) + 9(4-t) = 25\checkmark$$

and

$$7(3-t) + (4-2t) + 9t = 25\checkmark.$$

5. **(8 points) The temperature of a plate at the point (x, y) is $T(x, y) = 100 + x^2 - y^2$. Find the path that a heat seeking particle would travel if it starts at the point $(5, \sqrt{75})$. (The particle always moves in the direction of the greatest increase in temperature.)**

Let $\overrightarrow{\boldsymbol{r}}(t) = x(t)\overrightarrow{\boldsymbol{i}} + y(t)\overrightarrow{\boldsymbol{j}}$ give the position of the object at time t. So, the velocity of the object at time t is $\overrightarrow{\boldsymbol{r}}'(t) = x'(t)\overrightarrow{\boldsymbol{i}} + y'(t)\overrightarrow{\boldsymbol{j}}$. The velocity vector points in the direction of motion at time t. On the other hand, at every moment the direction of motion is perpendicular to whatever level set happens to contain the particle. Gradients are perpendicular to level sets. We know $\overrightarrow{\nabla} T = 2x\overrightarrow{\boldsymbol{i}} - 2y\overrightarrow{\boldsymbol{j}}$. So, the velocity vector of the particle at time t is a multiple of $\overrightarrow{\nabla} T$ at the point $(x(t), y(t))$. In other words,

$$x'(t)\overrightarrow{\boldsymbol{i}} + y'(t)\overrightarrow{\boldsymbol{j}} = c(t)(2x(t)\overrightarrow{\boldsymbol{i}} - 2y(t)\overrightarrow{\boldsymbol{j}}).$$

We solve the system of differential equations:

$$x'(t) = 2c(t)x(t) \quad y'(t) = -2c(t)y(t).$$

This is the same as

$$\frac{x'(t)}{x(t)} = 2c(t) \quad \text{and} \quad \frac{y'(t)}{y(t)} = -2c(t).$$

Integrate both sides with respect to t:

$$\int \frac{x'(t)}{x(t)}dt = \int 2c(t)dt \quad \text{and} \quad \int \frac{y'(t)}{y(t)}dt = \int -2c(t)dt,$$

to get

$$\ln|x(t)| = 2C(t) + c_1 \quad \text{and} \quad \ln|y(t)| = -2C(t) + c_2$$

where $C(t)$ is any fixed anti-derivative of $c(t)$. Exponentiate to see

$$|x(t)| = e^{c_1}\left(e^{C(t)}\right)^2 \quad \text{and} \quad |y(t)| = e^{c_2}\left(e^{C(t)}\right)^{-2}.$$

So

$$x(t) = \pm e^{c_1}\left(e^{C(t)}\right)^2 \quad \text{and} \quad y(t) = \pm e^{c_2}\left(e^{C(t)}\right)^{-2}.$$

Let K_1 be the constant $\pm e^{c_1}$, K_2 be the constant $\pm e^{c_2}$, and $\mathcal{C}(t)$ be the function $\left(e^{C(t)}\right)^2$. We have

$$x(t) = K_1\mathcal{C}(t) \quad \text{and} \quad y(t) = \frac{K_2}{\mathcal{C}(t)}.$$

Eliminate the parameter by using

$$\mathcal{C}(t) = \frac{K_2}{y(t)}$$

to see that the path of the object is

$$x(t) = K_1 \frac{K_2}{y(t)}.$$

In other words, the path of the object is $xy = K$, where K is the constant K_1K_2. The point $(5, \sqrt{75})$ is on the path of the object, so $5\sqrt{75} = K$ and the path of the object is $\boxed{xy = 5\sqrt{75}.}$

6.

(a) **Find** $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=3x}} \frac{x^3 y}{x^6+2y^2}$ **.**

We see that

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=3x}} \frac{x^3y}{x^6+2y^2} = \lim_{x\to 0} \frac{x^3(3x)}{x^6+2(3x)^2} = \lim_{x\to 0} \frac{3x^4}{x^6+18x^2} = \lim_{x\to 0} \frac{3x^4}{x^2(x^4+18)}$$

$$= \lim_{x\to 0} \frac{3x^2}{(x^4+18)} = \frac{0}{18} = \boxed{0.}$$

(b) **Find** $\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^3}} \frac{x^3 y}{x^6+2y^2}$ **.**

We see that

$$\lim_{\substack{(x,y)\to(0,0) \\ \text{along } y=x^3}} \frac{x^3y}{x^6+2y^2} = \lim_{x\to 0} \frac{x^3x^3}{x^6+2(x^3)^2} = \lim_{x\to 0} \frac{x^6}{x^6+2x^6} = \lim_{x\to 0} \frac{x^6}{3x^6} = \lim_{x\to 0} \frac{1}{3} = \boxed{\frac{1}{3}}.$$

7. **Let** $\overrightarrow{\boldsymbol{a}} = \overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}} + 3\overrightarrow{\boldsymbol{k}}$ **and** $\overrightarrow{\boldsymbol{b}} = 3\overrightarrow{\boldsymbol{i}} + 7\overrightarrow{\boldsymbol{j}} + 13\overrightarrow{\boldsymbol{k}}$**. Find vectors** $\overrightarrow{\boldsymbol{u}}$ **and** $\overrightarrow{\boldsymbol{v}}$ **with** $\overrightarrow{\boldsymbol{b}} = \overrightarrow{\boldsymbol{u}} + \overrightarrow{\boldsymbol{v}}$**,** $\overrightarrow{\boldsymbol{u}}$ **parallel to** $\overrightarrow{\boldsymbol{a}}$**, and** $\overrightarrow{\boldsymbol{v}}$ **perpendicular to** $\overrightarrow{\boldsymbol{a}}$**. Please check your answer.** We see that $\overrightarrow{\boldsymbol{u}}$ is the projection of $\overrightarrow{\boldsymbol{b}}$ onto $\overrightarrow{\boldsymbol{a}}$; so

$$\overrightarrow{\boldsymbol{u}} = \frac{\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{b}}}{\overrightarrow{\boldsymbol{a}} \cdot \overrightarrow{\boldsymbol{a}}} \overrightarrow{\boldsymbol{a}} = \frac{56}{14} \overrightarrow{\boldsymbol{a}} = 4(\overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}} + 3\overrightarrow{\boldsymbol{k}});$$

so

$$\boxed{\overrightarrow{\boldsymbol{u}} = 4\overrightarrow{\boldsymbol{i}} + 8\overrightarrow{\boldsymbol{j}} + 12\overrightarrow{\boldsymbol{k}}.}$$

We see that

$$\overrightarrow{\boldsymbol{v}} = \overrightarrow{\boldsymbol{b}} - \overrightarrow{\boldsymbol{u}} = 3\overrightarrow{\boldsymbol{i}} + 7\overrightarrow{\boldsymbol{j}} + 13\overrightarrow{\boldsymbol{k}} - (4\overrightarrow{\boldsymbol{i}} + 8\overrightarrow{\boldsymbol{j}} + 12\overrightarrow{\boldsymbol{k}}) = \boxed{-\overrightarrow{\boldsymbol{i}} - \overrightarrow{\boldsymbol{j}} + \overrightarrow{\boldsymbol{k}}.}$$

It is clear that $\overrightarrow{\boldsymbol{b}} = \overrightarrow{\boldsymbol{u}} + \overrightarrow{\boldsymbol{v}}$ and $\overrightarrow{\boldsymbol{u}}$ parallel to $\overrightarrow{\boldsymbol{a}}$. We check $\overrightarrow{\boldsymbol{v}} \cdot \overrightarrow{\boldsymbol{a}} = -1 - 2 + 3 = 0\checkmark$.