Math 241 Exam 1 Spring 2008 Solutions

Write your answers as legibly as you can on the blank sheets of paper provided. Use only **one side** of each sheet. Be sure to number your pages. Put your solution to problem 1 first, and then your solution to number 2, etc.; although, by using enough paper, you can do the problems in any order that suits you.

There are 6 problems. SHOW your work. Make your work be coherent and clear. Write in complete sentences whenever this is possible. $\boxed{CIRCLE}$ your answer. **CHECK** your answer whenever possible. **No Calculators.**

1. (8 points) **Find an equation for the line through the points** $P_1 = (5, -2, 1)$ **and** $P_2 = (2, 4, 2)$ **. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2} = <-3, 6, 1>$ and the answer is

$$\boxed{\overrightarrow{\boldsymbol{r}}(t) = <5 - 3t, -2 + 6t, 1 + t>}.$$

Check: The position vector at $t = 0$ is $\overrightarrow{\boldsymbol{r}}(0) = <5, -2, 1>$ and the position vector at time $t = 1$ is $\overrightarrow{\boldsymbol{r}}(1) = <2, 4, 2>$.

2. (8 points) **Find an equation for the plane through the points** $P_1 = (-2, 1, 1)$ **,** $P_2 = (0, 2, 3)$ **, and** $P_3 = (1, 0, 1)$ **. Check your answer. Make sure it is correct.**

We see that

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ 2 & 1 & 2 \\ 3 & -1 & 0 \end{vmatrix} = <2, 6, -5>$$

and the answer is

$$2(x + 2) + 6(y - 1) - 5(z - 1) = 0$$

or

$$\boxed{2x + 6y - 5z = -3.}$$

Check: The point P_1 satisfies our equation because

$$2(-2) + 6(1) - 5(1) = -3;$$

P_2 works:

$$2(0) + 6(2) - 5(3) = -3;$$

and P_3 works:

$$2(1) + 6(0) - 5(1) = -3.$$

3. (8 points) **Express** $\overrightarrow{\boldsymbol{v}} = <-2, 1, 6>$ **as the sum of a vector parallel to** $\overrightarrow{\boldsymbol{b}} = <0, -2, 1>$ **and a vector orthogonal to** $\overrightarrow{\boldsymbol{b}}$ **. Check your answer. Make sure it is correct.**

The projection of $\overrightarrow{\boldsymbol{v}}$ onto $\overrightarrow{\boldsymbol{b}}$ is

$$\text{proj}_{\overrightarrow{\boldsymbol{b}}}\, \overrightarrow{\boldsymbol{v}} = \frac{\overrightarrow{\boldsymbol{b}} \cdot \overrightarrow{\boldsymbol{v}}}{\overrightarrow{\boldsymbol{b}} \cdot \overrightarrow{\boldsymbol{b}}} \overrightarrow{\boldsymbol{b}} = \boxed{\overrightarrow{\boldsymbol{c}} = \tfrac{4}{5}<0, -2, 1>}$$

and

$$\overrightarrow{\boldsymbol{v}} - \text{proj}_{\overrightarrow{\boldsymbol{b}}}\, \overrightarrow{\boldsymbol{v}} = <-2, 1, 6> - \tfrac{4}{5}<0, -2, 1> = \boxed{\overrightarrow{\boldsymbol{d}} = <-2, \tfrac{13}{5}, \tfrac{26}{5}>}.$$

Check: It easy to see that $\overrightarrow{\boldsymbol{c}} + \overrightarrow{\boldsymbol{d}} = \overrightarrow{\boldsymbol{v}}$, $\overrightarrow{\boldsymbol{c}}$ is parallel to $\overrightarrow{\boldsymbol{b}}$ and $\overrightarrow{\boldsymbol{d}}$ is perpendicular to $\overrightarrow{\boldsymbol{b}}$.

4. (8 points) **A bowling ball of radius** R **is placed in a box just large enough to hold it, and is secured for shipping by packing a styrofoam sphere into each corner of the box. Find the radius of the largest styrofoam sphere that can be used.**

Draw a line from the center of the bowling ball to a corner of the box. (If the center of the bowling ball is the origin, then the corner of the box is (R, R, R) .) Your line has length $\sqrt{3}R$. Let r be the radius of a styrofoam ball. The same reasoning as above shows that the distance from the center of a styrofoam ball to a corner of the box is $\sqrt{3}r$. The line from the center of the bowling ball to the corner of the box is made up of one radius of the bowling ball, one radius of the styrofoam ball, and one trip from the center of the styrofoam ball to the corner of the box. (See the picture.) So $\sqrt{3}R = R + r + \sqrt{3}r$. We conclude that

$$r = \boxed{\frac{\sqrt{3}-1}{\sqrt{3}+1}R.}$$

5. (10 points) **Let** P **be the point** $P = (1, -2, 3)$ **and let** $\mathfrak{P}$ **be the plane** $2x - 2y + z = 4$ **.**
   a. **What is the distance from** P **to** $\mathfrak{P}$ **?**
   b. **What is the point on** $\mathfrak{P}$ **which is nearest to** P **?**
   c. **What is the equation of the line which is perpendicular to** $\mathfrak{P}$ **and passes through** P **?**

The answer to (c) is $\boxed{\overrightarrow{\boldsymbol{r}}(t) = <1+2t, -2-2t, 3+t>}$. The intersection of the answer to (c) and $\mathfrak{P}$ occurs when

$$2(1+2t) - 2(-2-2t) + (3+t) = 4;$$

so, $t = -\frac{5}{9}$ and the point on the plane closest to P is $\boxed{(-1/9, -8/9, 22/9)}$. The distance from P to $\mathfrak{P}$ is

$$\sqrt{\left(\frac{10}{9}\right)^2 + \left(\frac{10}{9}\right)^2 + \left(\frac{5}{9}\right)^2} = \frac{5}{9}\sqrt{4+4+1} = \boxed{\frac{5}{3}}.$$

Check: The distance from P to $\mathfrak{P}$ is

$$\left|\frac{2(1) - 2(-2) + 3 - 4}{\sqrt{4+4+1}}\right|$$

and this is also $\frac{5}{3}$.

6. (8 points) **Find the equation of the line tangent to** $\overrightarrow{\boldsymbol{r}}(t) = t\,\overrightarrow{\boldsymbol{i}} + t^2\,\overrightarrow{\boldsymbol{j}}$ **at the point** $P = (2,4)$**.**

We see $\overrightarrow{\boldsymbol{r}}'(t) = 1\,\overrightarrow{\boldsymbol{i}} + 2t\,\overrightarrow{\boldsymbol{j}}$, so $\overrightarrow{\boldsymbol{r}}'(2) = 1\,\overrightarrow{\boldsymbol{i}} + 4\,\overrightarrow{\boldsymbol{j}}$ and the line tangent to the curve $\overrightarrow{\boldsymbol{r}}$ at $t = 2$ is $\boxed{\overrightarrow{\ell}(t) = (2+t)\,\overrightarrow{\boldsymbol{i}} + (4+4t)\,\overrightarrow{\boldsymbol{j}}}$. Of course, this tangent line is also equal to $y = 4 + 4(x-2)$, which what you would have gotten if you used Math 141 techniques.