A vector projection problem that we did in class on Wednesday, Sept. 11, 2024

Write the vector $\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k}$ as the sum of two vectors, one of which is parallel to $\vec{b} = -\vec{i} + 4\vec{j} - 9\vec{k}$ and the other is perpendicular to $\vec{b}$.

Answer: The picture is on the next page. We write

$$\vec{v} = \mathrm{proj}_{\vec{b}}\,\vec{v} + (\vec{b} - \mathrm{proj}_{\vec{b}}\,\vec{v}),$$

where $\mathrm{proj}_{\vec{b}}\,\vec{v}$ is parallel to $\vec{b}$ and $\vec{b} - \mathrm{proj}_{\vec{b}}\,\vec{v}$ is perpendicular to $\vec{b}$.

We compute

$$\mathrm{proj}_{\vec{b}}\,\vec{v} = \frac{\vec{b}\cdot\vec{v}}{\vec{b}\cdot\vec{b}}\vec{b} = \frac{-1+8-27}{1+16+81}\vec{b} = \left(\frac{-20}{98}\right)(-\vec{i} + 4\vec{j} - 9\vec{k})$$
$$= \left(\frac{-10}{49}\right)(-\vec{i} + 4\vec{j} - 9\vec{k}).$$

We calculate

$$\vec{v} - \mathrm{proj}_{\vec{b}}\,\vec{v} = \vec{i} + 2\vec{j} + 3\vec{k} - \left(\frac{-10}{49}\right)(-\vec{i} + 4\vec{j} - 9\vec{k})$$
$$= \frac{1}{49}(39\vec{i} + 138\vec{j} + 57\vec{k}).$$

We conclude that

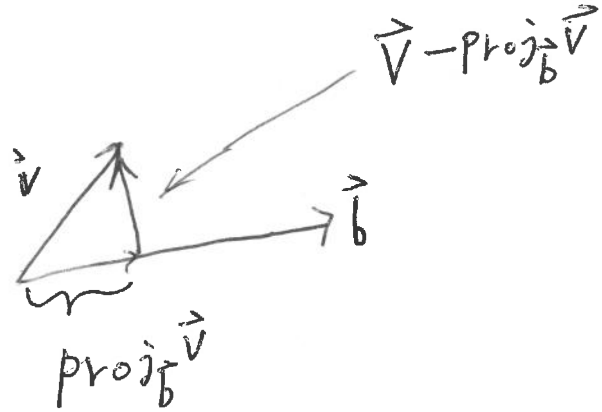
> The vector $\vec{v}$ is equal to
>
> $\frac{-10}{49}(-\vec{i} + 4\vec{j} - 9\vec{k})$ plus $\frac{1}{49}(39\vec{i} + 138\vec{j} + 57\vec{k})$
>
> with $\frac{-10}{49}(-\vec{i} + 4\vec{j} - 9\vec{k})$ parallel to $\vec{b}$
>
> and $\frac{1}{49}(39\vec{i} + 138\vec{j} + 57\vec{k})$ perpendicular to $\vec{v}$.

Check. We check all three assertions.

$$\frac{-10}{49}(-\vec{i} + 4\vec{j} - 9\vec{k}) + \frac{1}{49}(39\vec{i} + 138\vec{j} + 57\vec{k})$$
$$= \frac{1}{49}(49\vec{i} + 98\vec{j} + 147\vec{k}) = \vec{i} + 2\vec{j} + 3\vec{k}\checkmark$$

$\frac{-10}{49}(-\vec{i} + 4\vec{j} - 9\vec{k})$ is a multiple of $\vec{b}$. ✓

$$(\vec{i} + 4\vec{j} - 9\vec{k})\cdot\frac{1}{49}(39\vec{i} + 138\vec{j} + 57\vec{k})$$
$$= \frac{1}{49}(39 + 4(138) - 9(57)) = \frac{1}{49}(39 + 552 - 513) = 0.\checkmark$$



$\vec{v} - proj_{\vec{b}}\vec{v}$
$\vec{v}$
$\vec{b}$
$proj_{\vec{b}}\vec{v}$