Math 241, Exam 1, Fall, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find an equation for the plane through the points** $P_1 = (1, 2, 3)$**,** $P_2 = (1, 1, 1)$**, and** $P_3 = (-1, 0, 1)$**. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2}$ is equal to $-\vec{\boldsymbol{j}} - 2\vec{\boldsymbol{k}}$ and $\overrightarrow{P_1P_3} = -2\vec{\boldsymbol{i}} - 2\vec{\boldsymbol{j}} - 2\vec{\boldsymbol{k}}$. We compute

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 0 & -1 & -2 \\ -2 & -2 & -2 \end{vmatrix} = \begin{vmatrix} -1 & -2 \\ -2 & -2 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} 0 & -2 \\ -2 & -2 \end{vmatrix} \vec{\boldsymbol{j}} + \begin{vmatrix} 0 & -1 \\ -2 & -2 \end{vmatrix} \vec{\boldsymbol{k}}$$

$$= -2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - 2\vec{\boldsymbol{j}}.$$

The plane through $(1, 2, 3)$ perpendicular to $-2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - 2\vec{\boldsymbol{j}}$ is

$$-2(x - 1) + 4(y - 2) - 2(z - 3) = 0$$

or

$$-(x - 1) + 2(y - 2) - (z - 3) = 0$$

or

$$\boxed{-x + 2y - z = 0}.$$

Check. The point $(1, 2, 3)$ satisfies the proposed answer because

$$-1 + 2(2) - 3 = 0.$$

The point $(1, 1, 1)$ satisfies the proposed answer because

$$-1 + 2 - 1 = 0.$$

The point $(-1, 0, 1)$ satisfies the proposed answer because $-(-1) - 1 = 0$.

(2) **Express $\overrightarrow{v} = 4\overrightarrow{i} + 5\overrightarrow{j}$ as the sum of a vector parallel to $\overrightarrow{w} = 2\overrightarrow{i} + 5\overrightarrow{j}$ and a vector orthogonal to $\overrightarrow{w}$. Check your answer. Make sure it is correct.**

The projection of $\overrightarrow{v}$ onto $\overrightarrow{w}$ is

$$\text{proj}_{\overrightarrow{w}}\, \overrightarrow{v}\, \frac{\overrightarrow{v} \cdot \overrightarrow{w}}{\overrightarrow{w} \cdot \overrightarrow{w}} \overrightarrow{w} = \frac{33}{29}(2\overrightarrow{i} + 5\overrightarrow{j}) = \frac{66}{29}\overrightarrow{i} + \frac{165}{29}\overrightarrow{j}.$$

We compute

$$\overrightarrow{v} - \text{proj}_{\overrightarrow{w}}\, \overrightarrow{v} = (4\overrightarrow{i} + 5\overrightarrow{j}) - (\frac{66}{29}\overrightarrow{i} + \frac{165}{29}\overrightarrow{j})$$

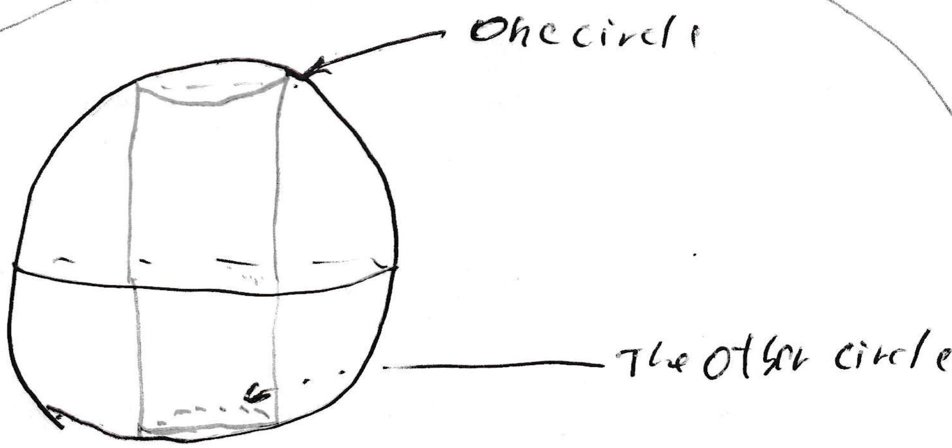
$$= \frac{1}{29}(116\overrightarrow{i} + 145\overrightarrow{j} - 66\overrightarrow{i} - 165\overrightarrow{j}) = \frac{1}{29}(50\overrightarrow{i} - 20\overrightarrow{j}) = \frac{10}{29}(5\overrightarrow{i} - 2\overrightarrow{j}).$$

Thus

$\overrightarrow{v} = \frac{33}{29}(2\overrightarrow{i} + 5\overrightarrow{j}) + \frac{10}{29}(5\overrightarrow{i} - 2\overrightarrow{j})$
with $\frac{33}{29}(2\overrightarrow{i} + 5\overrightarrow{j})$ parallel to $\overrightarrow{w}$
and $\frac{10}{29}(5\overrightarrow{i} - 2\overrightarrow{j})$ perpendicular to $\overrightarrow{w}$.

(3) **Name, describe, and graph the set of all points in three-space which satisfy both equations $x^2 + y^2 + z^2 = 4$ and $x^2 + y^2 = 1$.**

3 The graph of $x^2+y^2+z^2=4$ is the sphere with center $(0,0,0)$ and radius 2. The graph of $x^2+y^2=1$ is the cylinder with the z-axis in its center and radius 1.



The intersection of these 2 surfaces is 2 circles. Each circle has radius 1. The top circle has center $(0,0,\sqrt{3})$. This circle lives in the plane $z=\sqrt{3}$. The lower circle has center $(0,0,-\sqrt{3})$. This circle lives in the plane $z=-\sqrt{3}$.

(4) **Find the center and radius of the sphere** $9x^2 - 6x + 9y^2 + 6y + 9z^2 - 6z = 1$**.**

The solution set of $9x^2 - 6x + 9y^2 + 6y + 9z^2 - 6z = 1$ is the same as the solution set of

$$9(x^2 - \frac{2}{3}x + \boxed{\frac{1}{9}}) + 9(y^2 + \frac{2}{3}z + \boxed{\frac{1}{9}}) + 9(z^2 - \frac{2}{3}z + \boxed{\frac{1}{9}}) = 1 + 9\boxed{\frac{1}{9}} + 9\boxed{\frac{1}{9}} + 9\boxed{\frac{1}{9}}$$

$$9(x - \frac{1}{3})^2 + 9(y + \frac{1}{3})^2 + 9(z - \frac{1}{3})^2 = 4$$

$$(x - \frac{1}{3})^2 + (y + \frac{1}{3})^2 + (z - \frac{1}{3})^2 = \frac{4}{9}.$$

$$\boxed{\text{The sphere has center } (\frac{1}{3}, -\frac{1}{3}, \frac{1}{3}) \text{ and radius } \frac{2}{3}.}$$

(5) **Find the point on the plane** $2x + 3y - 7z = 1$ **which is closest to the point** $(-3, -4, 15)$**.**

The line perpendicular to the plane through $(-3, -4, 15)$ is

$$\begin{cases} x = -3 + 2t \\ y = -4 + 3t \\ z = 15 - 7t \end{cases}.$$

The line and the plane intersect when

$$2(-3 + 2t) + 3(-4 + 3t) - 7(15 - 7t) = 1$$

$$-6 + 4t - 12 + 9t - 105 + 49t = 1$$

$$62t = 124$$

$$t = 2$$

The point of intersection is $\boxed{(1, 2, 1)}$.

The proposed answer is on the plane because $2 + 6 - 7 = 1$. The vector from the proposed answer to $(-3, -4, 15)$ is $-4\vec{\boldsymbol{i}} - 6\vec{\boldsymbol{j}} + 14\vec{\boldsymbol{k}}$ which is indeed perpendicular to the plane.