Math 241, Exam 3, Fall, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **What is the equation of the plane tangent to $z = x^2 + y^2$ at the point where $x = 1$ and $y = 3$?**

The z-coordinate of the point on $z = x^2 + y^2$, when $x = 1$ and $y = 3$ is $1 + 9 = 10$.

View the surface $z = x^2 + y^2$ as a level set. It is $g(x, y, z) = 0$, where $g(x, y, z) = x^2 + y^2 - z$. Gradients are perpendicular to level sets. The gradient of g is $\overrightarrow{\nabla} g = 2x\,\overrightarrow{\boldsymbol{i}} + 2y\,\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}$. We calculate

$$\overrightarrow{\nabla} g|_(1, 3, 10) = 2\,\overrightarrow{\boldsymbol{i}} + 6\,\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}.$$

The plane tangent to $z = x^2 + y^2$ at the point $(1, 3, 10)$ is

$$\boxed{2(x - 1) + 6(y - 3) - (z - 10) = 0.}$$

(2) **Put $3x^2 + 2y^2 - 2z^2 - 12x - 4y + 12z = 8$ in the form**

$$A(x - x_0)^2 + B(y - y_0)^2 + C(z - z_0)^2 = D,$$

where x_0, y_0, z_0, A, B, C, and D are numbers.

Observe that the points that satisfy the original equation are exactly the same as the points which satisfy

$$3(x^2 - 4x + 4) + 2(y^2 - 2y + 1) - 2(z^2 - 6z + 9) = 8 + 12 + 2 - 18$$

$$\boxed{3(x - 2)^2 + 2(y - 1)^2 - 2(z - 3)^2 = 4.}$$

(3) **Consider the function** $f(x, y) = 9x^2 + 4y^2$ **and the point** $P = (1, 2)$**.**

(a) **Draw the level set** $f(x, y) = c$ **which contains the point** P**.**

(b) Calculate $\overrightarrow{\nabla} f|_P$**.**

(c) **Draw** $\overrightarrow{\nabla} f|_P$ **on your answer to (3a) with the tail on** P**.**

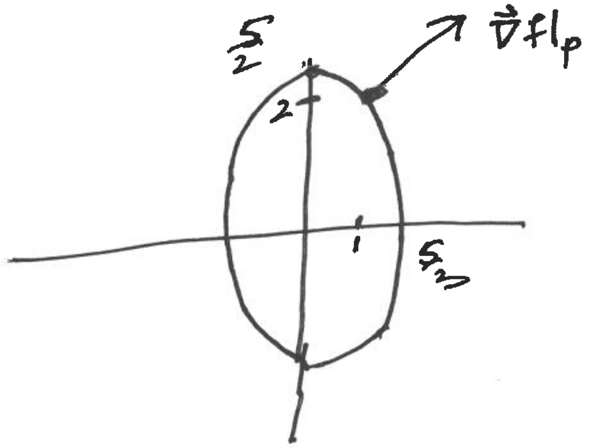
(d) **Calculate the directional derivative of the function** f **at the point** P **in the direction of the vector** $\overrightarrow{\boldsymbol{v}} = \overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}}$**.**

#3 $f(x,y) = 9x^2 + 4y^2 \quad P = (1,2)$

(a) $f(P) = 9(1)^2 + 4(2)^2 = 9 + 16 = 25$

$f(x,y) = 25$ is the ellipse $9x^2 + 4y^2 = 25$

$$\frac{x^2}{\frac{25}{9}} + \frac{y^2}{\frac{25}{4}} = 1$$



(b) $\vec{\nabla} f = 18x\vec{i} + 8y\vec{j}$

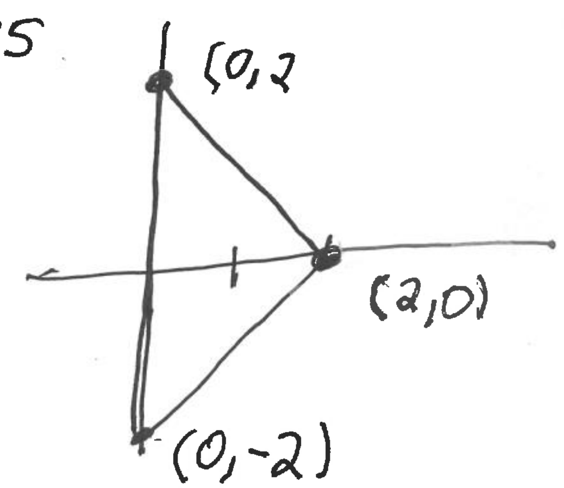
$\vec{\nabla} f|_p = 18\vec{i} + 16\vec{j}$

(c) is Drawn on (a) Notice that $\vec{\nabla} f|_p$ is perpendicular to $9x^2 + 4y^2 = 25$ at P.

(d) $\left(D_{\vec{i}+2\vec{j}} f\right)\Big|_p = \vec{\nabla} f\Big|_p \cdot \frac{\vec{i}+2\vec{j}}{\sqrt{5}} = (18\vec{i} + 16\vec{j}) \cdot \frac{\vec{i}+2\vec{j}}{\sqrt{5}}$

$= \boxed{\frac{18+32}{\sqrt{5}}}$

#5



(4) **Find all local maximum points, local minimum points, and saddle points of $f(x,y) = (y-2)x^2 - y^2$.**

We compute $f_x = 2x(y-2)$ and $f_y = x^2 - 2y$. Both partials are zero when $x = 0$ (and $y = 0$) and when $y = 2$ (and $x^2 = 4$). There are three such points, namely, $(0,0)$, $(2,2)$, and $(-2,2)$. We apply the second derivative test at each of these critical points.

$$f_{xx} = 2(y-2), \quad f_{xy} = 2x, \quad f_{yy} = -2.$$

We calculate the Hessian

$$H = f_{xx}f_{yy} - f_{xy}^2 = 2(y-2)(-2) - 4x^2 = -4y^2 + 8 - 4x^2.$$

We see that $H(0,0) = 8 > 0$ and $f_{xx}(0,0) = -4$, $H(2,2) < 0$ and $H(-2,2) < 0$. we conclude that

$(0,0,0)$ is a local maximum and $(2,2,-4)$ and $(-2,2,-4)$ are saddle points.

(5) **Find the absolute maximum points and absolute minimum points of $f(x,y) = x^2 + y^2 - 2x$ on the closed triangular region with vertices $(2,0)$, $(0,2)$, and $(0,-2)$.**

We drew the triangle on page 3. Of course, we save the three vertices $(2,0)$, $(0,2)$, and $(0,-2)$ as points of interest in the final step. We see that the boundary of the region is

$$x = 0 \text{ for } -2 \leq y \leq 2$$

$$y = x - 2 \text{ for } 0 \leq x \leq 2$$

$$y = -x + 2 \text{ for } 0 \leq x \leq 2.$$

We compute $f_x = 2x - 2$ and $f_y = 2y$. We see that $f_x = 0$ and $f_y = 0$ at $(x,y) = (1,0)$. The point $(1,0)$ is a point of interest.

The restriction of f to $x = 0$ is $f|_{x=0} = y^2$. We compute that $\frac{d}{dy}(f|_{x=0}) = 2y$ and this derivative vanishes at $(x,y) = (0,0)$. The point $(0,0)$ is a point of interest.

The restriction of f to $y = x - 2$ is $f|_{y=x-2} = x^2 + (x-2)^2 - 2x$. We compute that $\frac{d}{dy}(f|_{y=x-2}) = 2x + 2(x-2) - 2 = 4x - 6$, which is zero at $(x,y) = (\frac{3}{2}, -\frac{1}{2})$. The point $(\frac{3}{2}, -\frac{1}{2})$ is a point of interest.

The restriction of f to $y = -x + 2$ is $f|_{y=-x+2} = x^2 + (-x+2)^2 - 2x$. We compute that $\frac{d}{dy}(f|_{y=-x+2}) = 2x + 2(x-2) - 2 = 4x - 6$, which is zero at $(x,y) = (\frac{3}{2}, \frac{1}{2})$. The point $(\frac{3}{2}, \frac{1}{2})$ is a point of interest.

We calculate

$$\begin{aligned}
f(2,0) &= 0 \\
f(0,2) &= 4 \\
f(0,-2) &= 4 \\
f(1,0) &= -1 \\
f(0,0) &= 0 \\
f(\tfrac{3}{2},-\tfrac{1}{2}) &= -\tfrac{1}{2} \\
f(\tfrac{3}{2},\tfrac{1}{2}) &= -\tfrac{1}{2}
\end{aligned}$$

The maximum of f on the domain occurs at $(0,2,4)$ and $(0,-2,4)$.
The maximum of f on the domain occurs at $(1,0,-1)$.