Math 241, Exam 1, Fall, 2024 Solutions

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. (I will return your exam in the next class.) **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find a system of parametric equations for the line through the points $P_1 = (4, 5, 6)$ and $P_2 = (3, 2, 1)$. Check your answer. Make sure it is correct.**

The point $P = (x, y, z)$ is on the line precisely when

$$\overrightarrow{P_1P} = t\overrightarrow{P_1P_2},$$

for some t.

The point $P = (x, y, z)$ is on the line precisely when

$$(x-4)\overrightarrow{\boldsymbol{i}} + (y-5)\overrightarrow{\boldsymbol{j}} + (z-6)\overrightarrow{\boldsymbol{k}} = t(-\overrightarrow{\boldsymbol{i}} - 3\overrightarrow{\boldsymbol{j}} - 5\overrightarrow{\boldsymbol{k}}),$$

for some t.

The parametric equations for the line are

$$\begin{cases} x-4 = -t \\ y-5 = -3t \\ z-6 = -5t \end{cases}$$

The parametric equations for the line are

$$\boxed{\begin{cases} x = -t+4 \\ y = -3t+5 \\ z = 6-5t \end{cases}}$$

Check.

When $t = 0$ the line stands at $(-0+4, -3(0)+5, 6-5(0) = (4, 5, 6) = P_1.\checkmark$

When $t = 1$ the line stands at $(-1+4, -3(1)+5, 6-5(1)) = (3, 2, 1) = P_2.\checkmark$

(2) **Find an equation for the plane through the points $P_1 = (1,1,1)$, $P_2 = (-1,2,1)$, and $P_3 = (-2,1,2)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the plane. We compute

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -2 & 1 & 0 \\ -3 & 0 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} -2 & 0 \\ -3 & 1 \end{vmatrix} \vec{\boldsymbol{j}} + \begin{vmatrix} -2 & 1 \\ -3 & 0 \end{vmatrix} \vec{\boldsymbol{k}} \\ &= \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}.\end{aligned}$$

The point (x,y,z) is on the plane precisely when $\overrightarrow{P_1(x,y,z)}$ is perpendicular to $\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}$.
The point (x,y,z) is on the plane precisely when

$$\overrightarrow{(1,1,1)(x,y,z)} \cdot (\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 3\vec{\boldsymbol{k}}) = 0.$$

The equation of the plane is

$$(x-1) + 2(y-1) + 3(z-1) = 0$$

$$\boxed{x + 2y + 3z = 6}$$

Check.

We verify that P_1 satisfies the proposed answer:

$$1(1) + 2(1) + 3(1) = 6.\checkmark$$

We verify that P_2 satisfies the proposed answer:

$$1(-1) + 2(2) + 3(1) = 6.\checkmark$$

We verify that P_3 satisfies the proposed answer:

$$1(-2) + 2(1) + 3(2) = 6.\checkmark$$

(3) **Express $\vec{\boldsymbol{v}} = 2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}$ as the sum of a vector parallel to $\vec{\boldsymbol{b}} = \vec{\boldsymbol{i}} + \vec{\boldsymbol{j}}$ and a vector perpendicular to $\vec{\boldsymbol{b}}$. Check your answer. Make sure it is correct.**

The vector $\vec{\boldsymbol{v}}$ is equal to $\text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}}$ plus $\vec{\boldsymbol{v}} - \text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}}$ with $\text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}}$ parallel to $\vec{\boldsymbol{b}}$ and $\vec{\boldsymbol{v}} - \text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}}$ perpendicular to $\vec{\boldsymbol{b}}$ for

$$\text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{b}}}{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}}}\vec{\boldsymbol{b}}.$$

We calculate

$$\operatorname{proj}_{\vec{b}} \vec{v} = \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{2+4}{1+1} \vec{b} = 3\vec{b} = 3\vec{i} + 3\vec{j}.$$

We calculate

$$\vec{v} - \operatorname{proj}_{\vec{b}} \vec{v} = 2\vec{i} + 4\vec{j} - (3\vec{i} + 3\vec{j}) = -\vec{i} + \vec{j}.$$

We conclude that

$\vec{v} = (3\vec{i} + 3\vec{j}) + (-\vec{i} + \vec{j})$, with $3\vec{i} + 3\vec{j}$ parallel to $\vec{b}$ and $-\vec{i} + \vec{j}$ perpendicular to $\vec{b}$.

Check

Of course it is true that $(3\vec{i} + 3\vec{j}) + (-\vec{i} + \vec{j})$ is equal to $2\vec{i} + 4\vec{j}$, which is $\vec{v}$.✓

It is also true that $3\vec{i} + 3\vec{j}$ is parallel to $\vec{b} = \vec{i} + \vec{j}$.

We verify that $-\vec{i} + \vec{j}$ is perpendicular to $\vec{b}$:

$$(-\vec{i} + \vec{j}) \cdot (\vec{i} + \vec{j}) = -1 + 1 = 0\checkmark.$$

(4) **Name, describe, and graph the set of all points in three-space which satisfy $x^2 + z^2 = 1$. Is this object a finite set of points, or a curve, or a surface, or a solid?**

This object is a cylinder of radius 1 with the y-axis in its center. This object is a surface. $x^2 + z^2 = 1$ is the circle of radius 1 with center the origin in the xz-plane. We get the same picture for each y. The picture is on the last page.

(5) **Find the point on the plane $x + 2y + 3z = 10$ that is closest to the point $(5, 6, 7)$. Check your answer. Make sure it is correct.**

We find the line which passes through $(5, 6, 7)$ and is perpendicular to the plane. The intersection of the line that we found and the plane we were given is the answer.

The vector $\vec{N} = \vec{i} + 2\vec{j} + 3\vec{k}$ is perpendicular to the given plane. The parametric equations of the line through $(5, 6, 7)$ perpendicular to the given plane are

$$x - 5 = t, \quad y - 6 = 2t, \quad z - 7 = 3t.$$

The line and the plane meet when

$$(t + 5) + 2(2t + 6) + 3(3t + 7) = 10$$

$$14t + 38 = 10$$
$$14t = -28$$
$$t = -2$$

The point of intersection is $\boxed{(3, 2, 1)}$.

Check.

We see that the proposed answer is on the plane:

$$3 + 2(2) + 3(1) = 10, \checkmark$$

We verify that the vector from $(5, 6, 7)$ to $(3, 2, 1)$ is perpendicular to the plane. This vector is $-2\overrightarrow{\boldsymbol{i}} - 4\overrightarrow{\boldsymbol{j}} - 6\overrightarrow{\boldsymbol{k}} = -2\overrightarrow{\boldsymbol{N}}\checkmark$.

The picture of Problem 4