Math 241, Exam 3, Fall, 2024 Solutions

YOU SHOULD KEEP THIS PIECE OF PAPER. Write everything on the **blank paper provided**. Return the problems **IN ORDER** (use as much paper as necessary), use **ONLY ONE SIDE** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. (I will return your exam in the next class.) **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Consider the function** $f(x,y) = y - x^2$ **and the point** $P = (2,2)$**.**

(a) **Find the gradient of** f **at** P**.**

$\vec{\nabla} f = -2x\,\vec{i} + \vec{j}$; so $(\vec{\nabla} f)|_P = \boxed{-4\,\vec{i} + \vec{j}}$

(b) **Find the directional derivative of** f **in the direction of** $\vec{v} = \vec{i} + 2\vec{j}$ **at** P**.**

$D_{\vec{v}} f|_P = (\vec{\nabla} f)|_P \cdot \frac{\vec{v}}{|\vec{v}|} = (-4\,\vec{i} + \vec{j}) \cdot \frac{\vec{i}+2\vec{j}}{\sqrt{5}} = \boxed{\frac{-4+2}{\sqrt{5}}}$

(c) **Draw the level set of** f **that contains** P**.**

Observe that $f(P) = 2 - 4 = -2$. It follows that the level set which contains P is the graph of $y - x^2 = -2$; or $y = x^2 - 2$. We drew this on the last page.

(d) **Draw the gradient of** f **at** P**; put the tail of the gradient on** P**.**

We drew this on the last page.

(2) **Find the length of the curve** $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$**, for** $0 \le t \le \pi$**.**

The arc length is equal to

$$\int_0^\pi ||\vec{r}\,'(t)||dt = \int_0^\pi || -\sin t\,\vec{i} + \cos t\,\vec{j} + 1\,\vec{k}\,||dt$$
$$= \int_0^\pi \sqrt{\sin^2 t + \cos^2 t + 1}\,dt = \int_0^\pi \sqrt{2}dt = \boxed{\sqrt{2}\pi}.$$

(3) **Find all local maximum points, local minimum points, and saddle points of $f(x,y) = x^2y + 4xy - 2y^2$.**

We compute $f_x = 2xy + 4y$ and $f_y = x^2 + 4x - 4y$. We find all points, where f_x and f_y both vanish. We see that $f_x = 0$, when $2y(x+2) = 0$; so $x = -2$ or $y = 0$. Both partials vanish when either

$$x = -2 \text{ and } x^2 + 4x - 4y = 0 \qquad \text{OR} \qquad y = 0 \text{ and } x^2 + 4x - 4y = 0$$

So,

$$x = -2 \text{ and } 4 - 8 - 4y = 0 \qquad \text{OR} \qquad y = 0 \text{ and } x^2 + 4x = 0$$

So,

$$(x,y) = (-2,-1) \quad \text{or} \quad (0,0) \quad \text{or} \quad (-4,0).$$

We must do the second derivative test at each critical point.

We compute $f_{xx} = 2y$, $f_{xy} = 2x + 4$, $f_{yy} = -4$. We consider

$$D = f_{xx}f_{yy} - {f_{xy}}^2 = 2y(-4) - (2x+4)^2$$

We apply the second derivative test at $(-2,-1)$:

$$D|_{(-2,1)} = -8(-1) - 0$$

which is positive. $f_{xx}(-2,-1) = 2(-1)$ which is negative.
Thus, $(-2, 1, f(-2,1))$ is a local maximum.

We apply the second derivative test at $(0,0)$:

$$D|_{(0,0)} = -8(0) - 16$$

which is negative. Thus, $(0, 0, f(0,0))$ is a saddle point.

We apply the second derivative test at $(-4,0)$:

$$D|_{(-4,0)} = -8(0) - 16$$

which is negative. Thus, $(-4, 0, f(-4,0))$ is a saddle point.

We conclude that

$(-2, 1, f(-2,1))$ is a local maximum; $(0, 0, f(0,0))$ is a saddle point; and $(-4, 0, f(-4,0))$ is a saddle point.

(4) **Find the absolute maximum and absolute minimum values of the function**

$$f(x,y) = -x^2 - y^2 + 2x + 2y + 1$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 2 - x$.

The vertices of the region, namely $\underline{(0,0)}$, $\underline{(0,2)}$, and $\underline{(2,0)}$, all are points on interest.

We find all interior points where both partial derivatives vanish. We compute $f_x = -2x + 2$ and $f_y = -2y + 2$. Thus $f_x = 0$ and $f_y = 0$ at $\underline{(1,1)}$, which is another point of interest.

The restriction of f to $x = 0$ is $f|_{x=0} = -y^2 + 2y + 1$. We compute $\frac{d}{dy}(f|_{x=0}) = -2y + 2$. This derivative is zero at $(x, y) = \underline{(0,1)}$, which is a point of interest.

The restriction of f to $y = 0$ is $f|_{y=0} = -x^2 + 2x + 1$. We compute $\frac{d}{dx}(f|_{y=0}) = -2x + 2$. This derivative is zero at $(x, y) = \underline{(1,0)}$, which is a point of interest.

The restriction of f to $y = 2 - x$ is

$$f|_{y=2-x} = -x^2 - (2-x)^2 + 2x + 2(2-x) + 1 = -2x^2 + 4x + 1.$$

We compute $\frac{d}{dx}(f|_{y=2-x}) = -4x + 4$. This derivative is zero at $(x, y) = \underline{(1,1)}$, which is a point of interest. We compute

$$\begin{aligned} f(0,0) &= 1 \\ f(0,2) &= 1 \\ f(2,0) &= 1 \\ f(1,1) &= 3 \\ f(0,1) &= 2 \\ f(1,0) &= 2 \end{aligned}$$

The maximum of f on the domain occurs at $(1,1,3)$. The minimum of f on the domain occurs at $(0,0,1)$, $(0,2,1)$, and $(2,0,1)$.

(5) **Find the equation of the plane tangent to $z = x^2 + y^2$ at $(x, y) = (1, 2)$.**

The point on the graph of $z = x^2 + y^2$ of interest to us is $(1, 2, 5)$.

Gradients are perpendicular to level sets. We view $z = x^2 + y^2$ as the level set $f(x, y, z) = 0$, where f is the function $f(x, y, z) = x^2 + y^2 - z$. The gradient of f at $(1, 2, 5)$ is

$$(2x\vec{\boldsymbol{i}} + 2y\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}})|_{(1,2,5)} = 2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}.$$

The answer is the equation of the plane through the point $(1, 2, 5)$ perpendicular to $2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}$; which is

$$\boxed{2(x-1) + 4(y-2) - (z-5) = 0.}$$