Math 241, Exam 1, Spring, 2025, Solutions.

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. (I will return your exam in the next class.) **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the equation for the plane through the points** $P_1 = (2,1,1)$**,** $P_2 = (1,2,2)$**, and** $P_3 = (-1,3,2)$**. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the plane. We compute

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & 1 \\ -3 & 2 & 1 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} \vec{i} - \begin{vmatrix} -1 & 1 \\ -3 & 1 \end{vmatrix} \vec{j} + \begin{vmatrix} -1 & 1 \\ -3 & 2 \end{vmatrix} \vec{k}$$

$$= (1-2)\vec{i} - (-1+3)\vec{j} + (-2+3)\vec{k}$$

$$= -\vec{i} - 2\vec{j} + \vec{k}$$

The plane through $(2,1,1)$ perpendicular to $-\vec{i} - 2\vec{j} + \vec{k}$ is

$$-(x-2) - 2(y-1) + (z-1) = 0.$$

$$\boxed{-x - 2y + z = -3}$$

Check. The point $(2,1,1)$ is on our proposed answer because

$$-2 - 2(1) + 1 = -3. \checkmark$$

The point $(1,2,2)$ is on our proposed answer because

$$-1 - 2(2) + 2 = -3. \checkmark$$

The point $(-1,3,2)$ is on our proposed answer because

$$-(-1) - 2(3) + 2 = -3. \checkmark$$

(2) **Express $\vec{v} = 3\vec{i} - 2\vec{j} + 7\vec{k}$ as the sum of a vector parallel to $\vec{b} = 3\vec{i} - 6\vec{j} + 9\vec{k}$ and a vector perpendicular to $\vec{b}$. Check your answer. Make sure it is correct.**

We compute

$$\begin{aligned}\operatorname{proj}_{\vec{b}} \vec{v} &= \tfrac{\vec{v}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b} = \tfrac{9+12+63}{9+36+81}\vec{b} \\ &= \frac{84}{126}\vec{b} = \frac{7(12)}{7(18)}\vec{b} = \frac{2}{3}\vec{b} \\ &= 2\vec{i} - 4\vec{j} + 6\vec{k}.\end{aligned}$$

We also compute $\vec{v} - \operatorname{proj}_{\vec{b}} \vec{v} = 3\vec{i} - 2\vec{j} + 7\vec{k} - (2\vec{i} - 4\vec{j} + 6\vec{k}) = \vec{i} + 2\vec{j} + \vec{k}$.

We conclude that

> the vector $\vec{v}$ is equal to $2\vec{i} - 4\vec{j} + 6\vec{k}$ plus $\vec{i} + 2\vec{j} + \vec{k}$,
> with $2\vec{i} - 4\vec{j} + 6\vec{k}$ parallel to $\vec{b}$
> and $\vec{i} + 2\vec{j} + \vec{k}$ perpendicular to $\vec{b}$.

Check. We see that

- $2\vec{i} - 4\vec{j} + 6\vec{k}$ plus $\vec{i} + 2\vec{j} + \vec{k}$ equals $3\vec{i} - 2\vec{j} + 7\vec{k} = \vec{v}$,✓
- $2\vec{i} - 4\vec{j} + 6\vec{k}$ is a multiple of $3\vec{i} - 6\vec{j} + 9\vec{k} = \vec{b}$,✓and
- $(\vec{i} + 2\vec{j} + \vec{k})\cdot(3\vec{i} - 6\vec{j} + 9\vec{k}) = 3 - 12 + 9 = 0$.✓

(3) **Name, describe, and graph the set of all points in three-space which satisfy $z = y^2$. Is this object a finite set of points, or a curve, or a surface, or a solid?**

The graph is a surface. It is called a parabolic cylinder. Draw the parabola $z = y^2$ in the yz-plane, then draw the same picture for each plane parallel to the yz-plane. There is a picture on the last page.

(4) **Find the center and radius of the sphere $2x^2 + 2y^2 + 2z^2 - 4x + 8y - 8z = 32$.**

The given equation has the same solution set as

$$2(x^2-2x+\boxed{1})+2(y^2+4y+\boxed{\boxed{4}})+2(z^2-4z+\boxed{\boxed{\boxed{4}}}) = 32+2\boxed{1}+2\boxed{\boxed{4}}+2\boxed{\boxed{\boxed{4}}}$$

$$2(x-1)^2 + 2(y+2)^2 + 2(z-2)^2 = 50$$

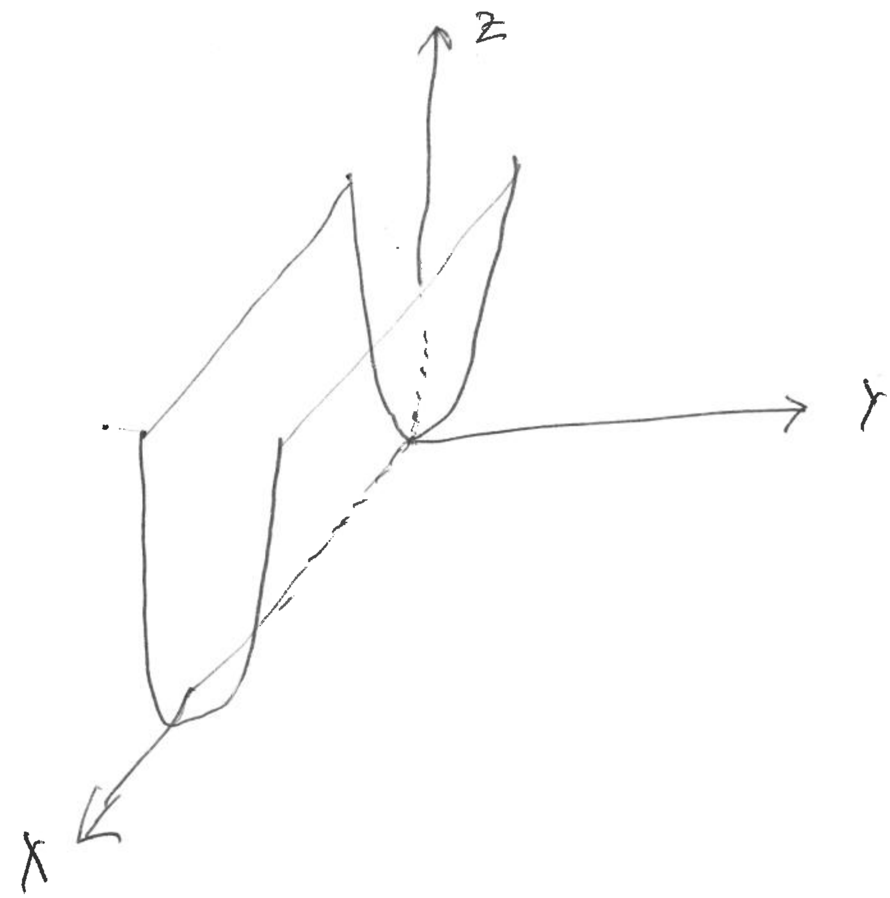
$$(x-1)^2 + (y+2)^2 + (z-2)^2 = 25$$

> The sphere has center $(1, -2, 2)$ and radius 5.

(5) **Give the equation, or equations, for the cylinder of radius 1 which has the y-axis in its center.**

The set of all points in 3-space which satisfy $\boxed{x^2 + z^2 = 1}$ form the cylinder of radius 1 with the y-axis in its center.

Picture for Problem 3



The set of points in 3-space which satisfy $z = y^2$ forms a parabolic cylinder.