Math 241, Exam 2, Fall, 2024, Solutions

YOU SHOULD KEEP THIS PIECE OF PAPER. Write everything on the **blank paper provided**. Return the problems **IN ORDER** (use as much paper as necessary), use **ONLY ONE SIDE** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. (I will return your exam in the next class.) **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find a system of parametric equations for the line tangent to the curve $\overrightarrow{r}(t) = e^t\,\overrightarrow{i} + 3t^2\,\overrightarrow{j} + \cos\frac{\pi t}{2}\,\overrightarrow{k}$ at $t = 1$.**

We compute

$$\begin{aligned}\overrightarrow{r}(1) &= e\,\overrightarrow{i} + 3\,\overrightarrow{j} + 0\,\overrightarrow{k},\\ \overrightarrow{r}'(t) &= e^t\,\overrightarrow{i} + 6t\,\overrightarrow{j} - \tfrac{\pi}{2}\sin\tfrac{\pi t}{2}, \text{ and}\\ \overrightarrow{r}'(1) &= e\,\overrightarrow{i} + 6\,\overrightarrow{j} - \tfrac{\pi}{2}\,\overrightarrow{k}.\end{aligned}$$

The line passing through $(e, 3, 0)$ parallel to $e\,\overrightarrow{i} + 6\,\overrightarrow{j} - \frac{\pi}{2}\,\overrightarrow{k}$ is

$$\boxed{x = e + et, \quad y = 3 + 6t, \quad z = -\tfrac{\pi}{2}t}.$$

(2) **Find an equation for the plane through the points $P_1 = (3, -2, 0)$, $P_2 = (7, 2, 1)$, and $P_3 = (2, 0, 2)$. Check your answer. Make sure it is correct.**

Observe that

$$\overrightarrow{P_1P_2} = 4\,\overrightarrow{i} + 4\,\overrightarrow{j} + \overrightarrow{k} \quad \text{and} \quad \overrightarrow{P_1P_3} = -\,\overrightarrow{i} + 2\,\overrightarrow{j} + 2\,\overrightarrow{k}.$$

It follows that

$$\begin{aligned}\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 4 & 4 & 1 \\ -1 & 2 & 2 \end{vmatrix}\\ &= \begin{vmatrix} 4 & 1 \\ 2 & 2 \end{vmatrix}\overrightarrow{i} - \begin{vmatrix} 4 & 1 \\ -1 & 2 \end{vmatrix}\overrightarrow{j} + \begin{vmatrix} 4 & 4 \\ -1 & 2 \end{vmatrix}\overrightarrow{k}\\ &= 6\,\overrightarrow{i} - 9\,\overrightarrow{j} + 12\,\overrightarrow{k}.\end{aligned}$$

The plane through $(3,-2,0)$ perpendicular to $6\vec{i}-9\vec{j}+12\vec{k}$ is

$$6(x-3)-9(y+2)+12(z-0)=0$$
$$2(x-3)-3(y+2)+4z=0$$
$$\boxed{2x-3y+4z=12.}$$

Check

Plug $(3,-2,0)$ into the proposed answer: $6+6+0=12\checkmark$.

Plug $(7,2,1)$ into the proposed answer: $14-6+4=12\checkmark$.

Plug $(2,0,2)$ into the proposed answer: $4-0+8=12\checkmark$.

(3) **Express $\vec{v}=3\vec{i}+\vec{j}$ as the sum of a vector parallel to $\vec{b}=4\vec{i}+8\vec{j}$ plus a vector perpendicular to $\vec{b}$. Check your answer. Make sure it is correct.**

There is a picture on the last page. We compute

$$\begin{aligned}\text{proj}_{\vec{b}}\,\vec{v}&=\frac{\vec{v}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\vec{b}\\&=\frac{12+8}{16+64}(4\vec{i}+8\vec{j})=\frac{1}{4}(4\vec{i}+8\vec{j})=\vec{i}+2\vec{j}.\end{aligned}$$

Observe that

$$\vec{v}-(\vec{i}+2\vec{j})=2\vec{i}-\vec{j}.$$

The vector $\vec{v}=3\vec{i}+\vec{j}$ is equal to $\vec{i}+2\vec{j}$ plus $2\vec{i}-\vec{j}$ with $\vec{i}+2\vec{j}$ parallel to $\vec{b}=4\vec{i}+8\vec{j}$ and $2\vec{i}-\vec{j}$ perpendicular to $\vec{b}=4\vec{i}+8\vec{j}$.

Check. We verify the three assertions:

$(\vec{i}+2\vec{j})+(2\vec{i}-\vec{j})=3\vec{i}+\vec{j}\checkmark$

$(\vec{i}+2\vec{j})=\frac{1}{4}(4\vec{i}+8\vec{j})\checkmark$

$(2\vec{i}-\vec{j})\cdot(4\vec{i}+8\vec{j})=8-8=0\checkmark$

(4) **The position vector at time t of an object moving on the xy-plane is $\vec{r}(t)=x(t)\vec{i}+y(t)\vec{j}$. If**

$$\vec{r}''(t)=2\vec{i},\quad \vec{r}'(0)=3\vec{i}+2\vec{j},\quad \textbf{and}\quad \vec{r}(0)=4\vec{i},$$

then what is the x-coordinate of the object when the y-coordinate is 4?

Integrate to learn
$$\vec{\boldsymbol{r}}'(t) = 2t\,\vec{\boldsymbol{i}} + \vec{c_1}.$$
Plug in $t = 0$ to see that
$$3\,\vec{\boldsymbol{i}} + 2\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{r}}'(0) = \vec{c_1}.$$
Thus,
$$\vec{\boldsymbol{r}}'(t) = (2t+3)\,\vec{\boldsymbol{i}} + 2\,\vec{\boldsymbol{j}}.$$
Integrate again to learn
$$\vec{\boldsymbol{r}}(t) = (t^2+3t)\,\vec{\boldsymbol{i}} + 2t\,\vec{\boldsymbol{j}} + \vec{c_2}.$$
Plug in $t = 0$ to see that
$$4\,\vec{\boldsymbol{i}} = \vec{\boldsymbol{r}}(0) = \vec{c_2}.$$
Thus,
$$\vec{\boldsymbol{r}}(t) = (t^2+3t+4)\,\vec{\boldsymbol{i}} + 2t\,\vec{\boldsymbol{j}}.$$
The y-coordinate of the object is 4 when $2t = 4$; so $t = 2$. When $t = 2$, then the x-coordinate of the object is $4 + 6 + 4 = \boxed{14}$.

(5) **Name, describe, and graph the set of all points in three-space which satisfy $y^2 + x^2 - z^2 = 1$. Is this object a finite set of points, or a curve, or a surface, or a solid?**

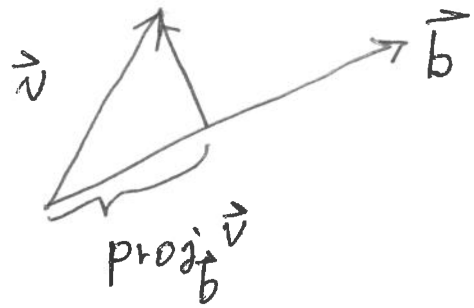
When $x = 0$ the equation becomes $y^2 - z^2 = 1$. This is a hyperbola which includes the points (y, z) equals $(1, 0)$ and $(-1, 0)$.

When $y = 0$ the equation becomes $x^2 - z^2 = 1$. This is a hyperbola which includes the points (x, z) equals $(1, 0)$ and $(-1, 0)$.

When $z = c$, for any constant c, the equation becomes $y^2 + x^2 = 1 + c^2$. This is a circle.

The equation describes a hyperboloid of one sheet, with the z-axis in its center. This is a surface. The surface may be obtained by drawing the hyperbola $y^2 - z^2 = 1$ in the yz-plane and then rotating the hyperbola about the z-axis. There is a picture on the next page.

Picture for number 3



Picture for number 5