12.3 This problem is problem 2, from the Final Exam from Fall 2024. **Express $\overrightarrow{v} = \overrightarrow{i} + 7\overrightarrow{j}$ as the sum of a vector parallel to $\overrightarrow{b} = \overrightarrow{i} + 2\overrightarrow{j}$ and a vector perpendicular to $\overrightarrow{b}$. Please make sure that your answer is correct.**

Answer: There is a picture on the next page. We compute

$$\begin{aligned}
\operatorname{proj}_{\overrightarrow{b}} \overrightarrow{v} &= \frac{\overrightarrow{v} \cdot \overrightarrow{b}}{\overrightarrow{b} \cdot \overrightarrow{b}} \overrightarrow{b} \\
&= \frac{(\overrightarrow{i} + 7\overrightarrow{j}) \cdot (\overrightarrow{i} + 2\overrightarrow{j})}{(\overrightarrow{i} + 2\overrightarrow{j}) \cdot (\overrightarrow{i} + 2\overrightarrow{j})} (\overrightarrow{i} + 2\overrightarrow{j}) \\
&= \frac{15}{5} (\overrightarrow{i} + 2\overrightarrow{j}) \\
&= 3\overrightarrow{i} + 6\overrightarrow{j}.
\end{aligned}$$

We also compute

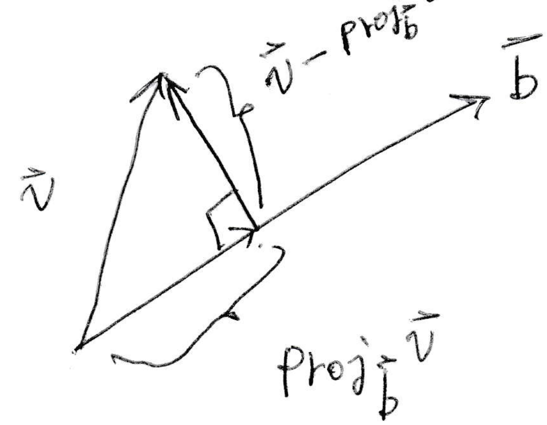
$$\overrightarrow{v} - \operatorname{proj}_{\overrightarrow{b}} \overrightarrow{v} = (\overrightarrow{i} + 7\overrightarrow{j}) - (3\overrightarrow{i} + 6\overrightarrow{j}) = -2\overrightarrow{i} + \overrightarrow{j}.$$

We conclude that

$\overrightarrow{v} = (3\overrightarrow{i} + 6\overrightarrow{j}) + (-2\overrightarrow{i} + \overrightarrow{j})$, with $3\overrightarrow{i} + 6\overrightarrow{j}$ parallel to $\overrightarrow{b}$ and $-2\overrightarrow{i} + \overrightarrow{j}$ perpendicular to $\overrightarrow{b}$

Check. Observe that

- $(3\overrightarrow{i} + 6\overrightarrow{j}) + (-2\overrightarrow{i} + \overrightarrow{j}) = 1\overrightarrow{i} + 7\overrightarrow{j} = \overrightarrow{v}\checkmark$,
- $3\overrightarrow{i} + 6\overrightarrow{j}$, which is equal to 3 times $\overrightarrow{b}$ is parallel to $\overrightarrow{b}$ $\checkmark$, and
- $(-2\overrightarrow{i} + \overrightarrow{j}) \cdot \overrightarrow{b} = (-2\overrightarrow{i} + \overrightarrow{j}) \cdot (\overrightarrow{i} + 2\overrightarrow{j}) = 0.\checkmark$


$\vec{v}$
$\vec{v} - \text{proj}_{\vec{b}}\vec{v}$
$\vec{b}$
$\text{proj}_{\vec{b}}\vec{v}$