Math 241, Final Exam, Spring, 2023

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 100 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the equation of the plane that contains the points** $(0, 1, 2)$**,** $(-1, 2, 3)$**, and** $(-4, -1, 2)$**. DEMONSTRATE that your answer is correct.**

Let $P = (0, 1, 2)$, $Q = (-1, 2, 3)$, and $R = (-4, -1, 2)$. We see that $\overrightarrow{PQ} = -\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} + \overrightarrow{\boldsymbol{k}}$ and $\overrightarrow{PR} = -4\overrightarrow{\boldsymbol{i}} - 2\overrightarrow{\boldsymbol{j}}$. Thus,

$$\overrightarrow{PQ} \times \overrightarrow{PR} = \det \begin{bmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ -1 & 1 & 1 \\ -4 & -2 & 0 \end{bmatrix}$$

$$= \det \begin{bmatrix} 1 & 1 \\ -2 & 0 \end{bmatrix} \overrightarrow{\boldsymbol{i}} - \det \begin{bmatrix} -1 & 1 \\ -4 & 0 \end{bmatrix} \overrightarrow{\boldsymbol{j}} + \det \begin{bmatrix} -1 & 1 \\ -4 & -2 \end{bmatrix} \overrightarrow{\boldsymbol{k}}$$

$$= 2\overrightarrow{\boldsymbol{i}} - 4\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}.$$

The plane through $(0, 1, 2)$ perpendicular to $2\overrightarrow{\boldsymbol{i}} - 4\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}$ is

$$2(x - 0) - 4(y - 1) + 6(z - 2) = 0.$$

Our answer is the same as

$$x - 2(y - 1) + 3(z - 2) = 0$$

or

$$\boxed{x - 2y + 3z = 4.}$$

Check

$$\begin{aligned} 0 - 2(1) + 3(2) &= 4 \\ -1 - 2(2) + 3(3) &= 4 \\ -4 - 2(-1) + 3(2) &= 4 \end{aligned}$$

(2) **Express $\overrightarrow{v} = 3\overrightarrow{i} + 5\overrightarrow{j} + \overrightarrow{k}$ as the sum of a vector parallel to $\overrightarrow{w} = \overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$ and a vector perpendicular to $\overrightarrow{w}$. DEMONSTRATE that your answer is correct.**

There is a picture at the end of the answer sheet. We calculate

$$\text{proj}_{\overrightarrow{w}}\, \overrightarrow{v} = \frac{\overrightarrow{w} \cdot \overrightarrow{v}}{\overrightarrow{w} \cdot \overrightarrow{w}} \overrightarrow{w} = \frac{3 + 10 - 1}{1 + 4 + 1}(\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k})$$
$$= \frac{12}{6}(\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}) = 2\overrightarrow{i} + 4\overrightarrow{j} - 2\overrightarrow{k}.$$

We see that

$$\overrightarrow{v} - \text{proj}_{\overrightarrow{w}}\, \overrightarrow{v} = 3\overrightarrow{i} + 5\overrightarrow{j} + \overrightarrow{k} - (2\overrightarrow{i} + 4\overrightarrow{j} - 2\overrightarrow{k})$$
$$= \overrightarrow{i} + \overrightarrow{j} + 3\overrightarrow{k}.$$

We conclude that

$\overrightarrow{v} = (2\overrightarrow{i} + 4\overrightarrow{j} - 2\overrightarrow{k}) + (\overrightarrow{i} + \overrightarrow{j} + 3\overrightarrow{k})$
with $2\overrightarrow{i} + 4\overrightarrow{j} - 2\overrightarrow{k}$ parallel to $\overrightarrow{w}$
and $\overrightarrow{i} + \overrightarrow{j} + 3\overrightarrow{k}$ perpendicular to $\overrightarrow{w}$.

Check. It is clear that $(2\overrightarrow{i} + 4\overrightarrow{j} - 2\overrightarrow{k}) + (\overrightarrow{i} + \overrightarrow{j} + 3\overrightarrow{k}) = 3\overrightarrow{i} + 5\overrightarrow{j} + 1\overrightarrow{k}$. It is clear that $2\overrightarrow{i} + 4\overrightarrow{j} - 2\overrightarrow{k}$ is parallel to $\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}$. We compute $(\overrightarrow{i} + \overrightarrow{j} + 3\overrightarrow{k}) \cdot (\overrightarrow{i} + 2\overrightarrow{j} - \overrightarrow{k}) = 1 + 2 - 3 = 0.$✓

(3) **Find the maximum of $f = 49 - x^2 - y^2$ on the line $x + 3y = 10$.**

We use the method of Lagrange multipliers and find all points on $x+3y = 10$ where $\overrightarrow{\nabla} f = \lambda \overrightarrow{\nabla} g$, for some λ, where $g = x + 3y$. Notice that for points on $x + 3y = 10$ which are far from the origin, f is very small. It is fortunate that we are asked to find only the maximum of f on $x + 3y = 10$ because f does not have a minimum on $x + 3y = 10$. At any rate $\overrightarrow{\nabla} f = -2x\overrightarrow{i} - 2y\overrightarrow{j}$ and $\overrightarrow{\nabla} g = \overrightarrow{i} + 3\overrightarrow{j}$. We look for all points, where

$$\begin{cases} x + 3y = 10 \\ -2x\overrightarrow{i} - 2y\overrightarrow{j} = \lambda(\overrightarrow{i} + 3\overrightarrow{j}) \end{cases} \quad \text{or} \quad \begin{cases} x + 3y = 10 \\ -2x = \lambda \\ -2y = 3\lambda \end{cases}$$

Read equation 2 to say $x = \lambda/(-2)$ and equation 3 to say that $y = 3\lambda/(-2)$. Now equation 1 says $\lambda/(-2) + 3(3\lambda)/(-2) = 10$ or $10\lambda = -20$, so $\lambda = -2$ and $x = 1$ and $y = 3$.

The maximum of f on $x + 3y = 10$ occurs at $(1, 3)$.
The maximum value of f on $x + 3y = 10$ is 39.

(4) **Find the volume between $z = 2 - x^2 - y^2$ and $z = x^2 + y^2 - 2$. (You must draw a meaningful picture.)**

There is a picture at the end of the answer sheet. The intersection is the circle $x^2 + y^2 = 2$ in the xy-plane. We integrate top-bottom over the intersection. The volume is

$$\int_0^{2\pi} \int_0^{\sqrt{2}} ((2 - r^2) - (r^2 - 2))r\, dr\, d\theta$$

$$= \int_0^{2\pi} \int_0^{\sqrt{2}} 4r - 2r^3\, dr\, d\theta$$

$$= \int_0^{2\pi} 2r^2 - \frac{r^4}{2}\Big|_0^{\sqrt{2}} d\theta$$

$$= 2\pi(4 - 2) = \boxed{4\pi}.$$

(5) **Compute $\int_0^1 \int_0^{\sqrt{1-x^2}} e^{x^2+y^2}\, dy\, dx$.**

We do the problem in polar coordinates. We are integrating over the quarter of the unit circle which is in the first quadrant.

$$\int_0^{\pi/2} \int_0^1 re^{r^2} dr\, d\theta = (\pi/2)\frac{1}{2}e^{r^2}\Big|_0^1$$

$$= \boxed{\frac{\pi}{4}(e - 1)}.$$

(6) **Find the absolute extreme points of $f(x, y) = 2 + 2x + 4y - x^2 - y^2$ on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.**

There is a picture at the end of the answer sheet. The three corners $(0, 0)$, $(9, 0)$, and $(0, 9)$ are points of interest.

We calculate the interior points where both partial derivatives are zero: $f_x = 2 - 2x$ and $f_y = 4 - 2y$. Both partial derivatives are zero at $(1, 2)$, which is in our domain. This point is a point of interest.

We look at f on the boundary $y = 0$: $f|_{y=0} = 2 + 2x - x^2$. We calculate $\frac{d(f|_{y=0})}{dx} = 2 - 2x$. This function is zero at $(1, 0)$, which is a point of interest.

We look at f on the boundary $x = 0$: $f|_{x=0} = 2 + 4y - y^2$. We calculate $\frac{d(f|_{x=0})}{dy} = 4 - 2y$. This function is zero at $(0, 2)$, which is a point of interest.

We look at f on the boundary $y = 9 - x$:

$$f|_{y=9-x} = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2.$$

We calculate $\frac{d(f|_{y=9-x})}{dx} = 2 - 4 - 2x + 2(9 - x) = 16 - 4x$. This function is zero at $(4, 5)$, which is a point of interest.

We evalute f at each point of interest:

$$\begin{aligned}
&f(0,0) = 2\\
&f(9,0) = 2 + 2(9) - 81 = -61\\
&f(0,9) = 2 + 4(9) - 81 = -43\\
&f(1,2) = 2 + 2 + 8 - 1 - 4 = 7\\
&f(1,0) = 2 + 2 - 1 = 3\\
&f(0,2) = 2 + 8 - 4 = 6\\
&f(4,5) = 2 + 8 + 20 - 16 - 25 = -11
\end{aligned}$$

The maximum of f on the given domain occurs at $(1, 2, 7)$.
The minimum of f on the given domain occurs at $(9, 0, -61)$.

(7) **Find the directional derivative of $f(x, y, z) = x^3 - xy^2 - z$ at the point $P = (1, 1, 0)$, in the direction of $\overrightarrow{\boldsymbol{v}} = 2\overrightarrow{\boldsymbol{i}} - 3\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}$.**

$$\begin{aligned}
(D_{\overrightarrow{\boldsymbol{v}}} f)|_P &= (\overrightarrow{\nabla} f)|_P \cdot \frac{\overrightarrow{\boldsymbol{v}}}{|\overrightarrow{\boldsymbol{v}}|}\\
&= ((3x^2 - y^2)\overrightarrow{\boldsymbol{i}} - 2xy\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}})_{(1,1,0)} \cdot \frac{2\overrightarrow{\boldsymbol{i}} - 3\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}}{\sqrt{4 + 9 + 36}}\\
&= (2\overrightarrow{\boldsymbol{i}} - 2\overrightarrow{\boldsymbol{j}} - \overrightarrow{\boldsymbol{k}}) \cdot \frac{2\overrightarrow{\boldsymbol{i}} - 3\overrightarrow{\boldsymbol{j}} + 6\overrightarrow{\boldsymbol{k}}}{7} = \frac{4 + 6 - 6}{7} = \boxed{\frac{4}{7}}.
\end{aligned}$$

(8) **Find the volume of the solid above the upper part of $x^2 + y^2 = 3z^2$ and below $x^2 + y^2 + z^2 = 1$.**

There is a picture at the end of the answer sheet. We find this volume in speherical coordinates. The picture demonstrates that the intersection occurs at $\phi = \frac{\pi}{3}$.

The volume is equal to

$$\begin{aligned}
&\int_0^{2\pi} \int_0^{\frac{\pi}{3}} \int_0^1 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta\\
&= \int_0^{2\pi} \int_0^{\frac{\pi}{3}} \frac{\rho^3}{3}\Big|_0^1 \sin\phi \, d\phi \, d\theta
\end{aligned}$$

$$= \int_0^{2\pi} -\frac{1}{3}\cos\phi\Big|_0^{\frac{\pi}{3}}\,d\theta$$

$$= \boxed{\tfrac{1}{3}(-\tfrac{1}{2}+1)2\pi.}$$

(9) **Find the local maxima, local minima, and saddle points of $f(x,y) = xy - x^2 - y^2 - 2x - 2y + 4$.**

We compute $f_x = y - 2x - 2$ and $f_y = x - 2y - 2$. Both partial derivatives are zero when $y = 2x+2$ and $0 = x - 2(2x+2) - 2$. So $3x = -6$, $x = -2$, and $y = -2$. The point $(-2,-2)$ is the only critical point. We apply the second derivative test. We see that $f_{xx} = -2$, $f_{xy} = 1$, and $f_{yy} = -2$. Thus, the Hessian, is

$$H = f_{xx}f_{yy} - f_{xy}^2 = 4 - 1 = 3 > 0$$

and $f_x x < 0$. We conclude that

$$\boxed{(-2,-2,f(-2,-2)) \text{ is a local maximum point of } z = f(x,y).}$$

(10) **Find the area of the region bounded by $y + x^2 = 2$ and $y + x = 0$. (You must draw a meaningful picture.)**

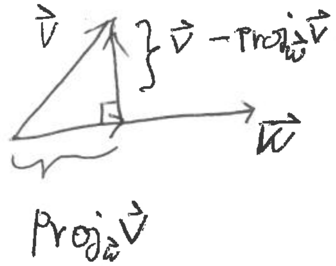
There is a picture at the end of the answer sheet.

Observe that $y = 2 - x^2$ is a parabola with with vertex at $(0,2)$ opening downward and $y = -x$ is the line through the origin with slope -1. These two curves intersect at $(2,-2)$ and $(-1,1)$. For each fixed x, with $-1 \le x \le 2$, y goes from $-x$ to $2 - x^2$. The area is

$$\int_{-1}^{2}\int_{-x}^{2-x^2} dy\,dx = \int_{-1}^{2}(2 - x^2 + x)\,dx = (2x - \tfrac{x^3}{3} + \tfrac{x^2}{2})\Big|_{-1}^{2}$$

$$= 4 - \tfrac{8}{3} + 2 - (-2 + \tfrac{1}{3} + \tfrac{1}{2}) = \boxed{\tfrac{9}{2}}.$$

2



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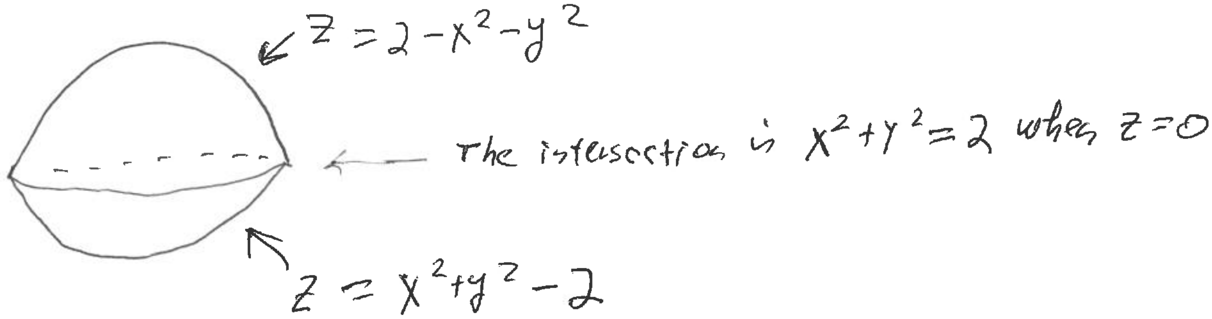

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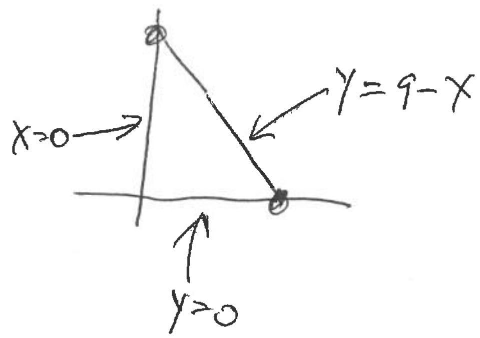

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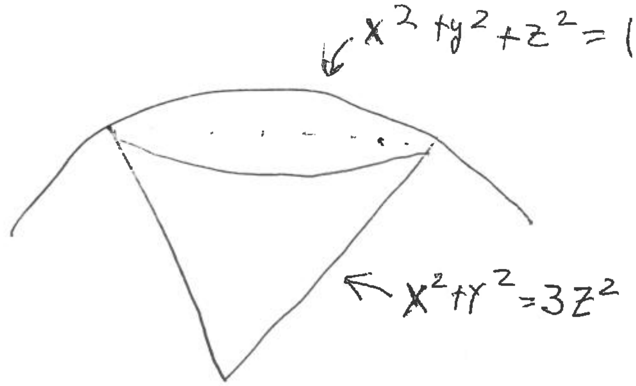
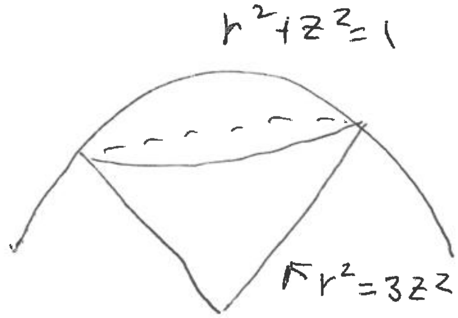
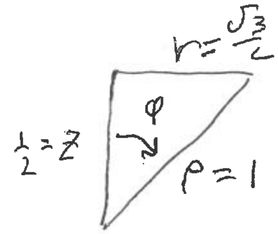




at the intersection $r^2+z^2=1$
$r^2=3z^2$

so $4z^2=1$
$z=\frac{1}{2}$
$r=\frac{\sqrt{3}}{2}$



$\cos\varphi = \frac{ADJ}{HYP} = \frac{1}{2}$ $\therefore$ $\varphi = \frac{\pi}{3}$

(10)