Math 241, Final Exam, Fall, 2024 Solutions

YOU SHOULD KEEP THIS PIECE OF PAPER. Write everything on the **blank paper provided**. Return the problems **IN ORDER** (use as much paper as necessary), use **ONLY ONE SIDE** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. **Fold your exam in half** before you turn it in.

The exam is worth 100 points. There are 10 problems; each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the equation of the plane that contains the points** $P = (1,1,1)$**,** $Q = (-3,1,-1)$**, and** $R = (-2,3,1)$**. Please make sure that your answer is correct.**

We compute

$$\begin{aligned}\overrightarrow{PQ} &= -4\,\overrightarrow{\boldsymbol{i}} + 0\,\overrightarrow{\boldsymbol{j}} - 2\,\overrightarrow{\boldsymbol{k}}\\ \overrightarrow{PR} &= -3\,\overrightarrow{\boldsymbol{i}} + 2\,\overrightarrow{\boldsymbol{j}} + 0\,\overrightarrow{\boldsymbol{k}}\end{aligned}$$

It follows that

$$\begin{aligned}\overrightarrow{PQ} \times \overrightarrow{PR} &= \begin{vmatrix} \overrightarrow{\boldsymbol{i}} & \overrightarrow{\boldsymbol{j}} & \overrightarrow{\boldsymbol{k}} \\ -4 & 0 & -2 \\ -3 & 2 & 0 \end{vmatrix}\\ &= \begin{vmatrix} 0 & -2 \\ 2 & 0 \end{vmatrix} \overrightarrow{\boldsymbol{i}} - \begin{vmatrix} -4 & -2 \\ -3 & 0 \end{vmatrix} \overrightarrow{\boldsymbol{j}} + \begin{vmatrix} -4 & 0 \\ -3 & 2 \end{vmatrix} \overrightarrow{\boldsymbol{k}}\\ &= 4\,\overrightarrow{\boldsymbol{i}} + 6\,\overrightarrow{\boldsymbol{j}} - 8\,\overrightarrow{\boldsymbol{j}}\end{aligned}$$

The plane through $(1,1,1)$ perpendicular to $4\,\overrightarrow{\boldsymbol{i}} + 6\,\overrightarrow{\boldsymbol{j}} - 8\,\overrightarrow{\boldsymbol{j}}$ is

$$4(x-1) + 6(y-1) - 8(z-1) = 0$$

$$2(x-1) + 3(y-1) - 4(z-1) = 0$$

$$\boxed{2x + 3y - 4z = 1}$$

Check. Plug each of the three points into the proposed answer:

$$\begin{array}{ll} P = (1,1,1): & 2(1) + 3(1) - 4(1) = 1\checkmark \\ Q = (-3,1,-1): & 2(-3) + 3(1) - 4(-1) = 1\checkmark \\ R = (-2,3,1): & 2(-2) + 3(3) - 4(1) = 1\checkmark \end{array}$$

(2) **Express $\vec{v} = \vec{i} + 7\vec{j}$ as the sum of a vector parallel to $\vec{b} = \vec{i} + 2\vec{j}$ and a vector perpendicular to $\vec{b}$. Please make sure that your answer is correct.**

We compute

$$\begin{aligned}
\operatorname{proj}_{\vec{b}} \vec{v} &= \frac{\vec{v} \cdot \vec{b}}{\vec{b} \cdot \vec{b}} \vec{b} \\
&= \frac{(\vec{i} + 7\vec{j}) \cdot (\vec{i} + 2\vec{j})}{(\vec{i} + 2\vec{j}) \cdot (\vec{i} + 2\vec{j})} (\vec{i} + 2\vec{j}) \\
&= \frac{15}{5} (\vec{i} + 2\vec{j}) \\
&= 3\vec{i} + 6\vec{j}.
\end{aligned}$$

We also compute

$$\vec{v} - \operatorname{proj}_{\vec{b}} \vec{v} = (\vec{i} + 7\vec{j}) - (3\vec{i} + 6\vec{j}) = -2\vec{i} + \vec{j}.$$

We conclude that

$\vec{v} = (3\vec{i} + 6\vec{j}) + (-2\vec{i} + \vec{j})$, with $3\vec{i} + 6\vec{j}$ parallel to $\vec{b}$ and $-2\vec{i} + \vec{j}$ perpendicular to $\vec{b}$

Check. Observe that

- $(3\vec{i} + 6\vec{j}) + (-2\vec{i} + \vec{j}) = 1\vec{i} + 7\vec{j} = \vec{v} \checkmark$,
- $3\vec{i} + 6\vec{j}$, which is equal to 3 times $\vec{b}$ is parallel to $\vec{b}$ $\checkmark$, and
- $(-2\vec{i} + \vec{j}) \cdot \vec{b} = (-2\vec{i} + \vec{j}) \cdot (\vec{i} + 2\vec{j}) = 0. \checkmark$

(3) **Consider the function $f(x, y) = x - y^2$ and the point $P = (2, 2)$.**

(a) **Find the gradient of f at P.**

$$\vec{\nabla} f|_P = (f_x \vec{i} + f_y \vec{j})|_P = (\vec{i} - 2y\vec{j})|_{(2,2)} = \boxed{\vec{i} - 4\vec{j}}.$$

(b) **Find the directional derivative of f in the direction of $\vec{v} = 3\vec{i} - \vec{j}$ at P.**

$$D_{\vec{v}} f|_P = \vec{\nabla} f|_P \cdot \frac{\vec{v}}{|\vec{v}|} = (\vec{i} - 4\vec{j}) \cdot \frac{3\vec{i} - \vec{j}}{\sqrt{10}} = \boxed{\frac{7}{\sqrt{10}}}.$$

(c) **Draw the level set of f that contains P.**

We see that $f(2,2) = 2 - 4 = -2$. The level set of f which contains P is $x - y^2 = -2$; which is the same as $x = y^2 - 2$. The graph is a parabola, knocked on its side (with the positive axis in its center) shifted to the left by 2. We drew a picture on the picture page, at the end of the solution set.

(d) **Draw the gradient of f at P; put the tail of the gradient on P.**

We drew this gradient on the picture page.

(4) **Find the length of the curve $\overrightarrow{\boldsymbol{r}}(t) = \cos 2t\,\overrightarrow{\boldsymbol{i}} + \sin 2t\,\overrightarrow{\boldsymbol{j}} + t\,\overrightarrow{\boldsymbol{k}}$, for $0 \le t \le \pi$.**

Arc length is the integral of speed; so the arc length is

$$\begin{aligned}
&\int_0^\pi |\overrightarrow{\boldsymbol{r}}'(t)|\,dt \\
=& \int_0^\pi |(-2\sin 2t)\,\overrightarrow{\boldsymbol{i}} + (2\cos 2t)\,\overrightarrow{\boldsymbol{j}} + (1)\,\overrightarrow{\boldsymbol{k}}\,|\,dt \\
=& \int_0^\pi \sqrt{4\sin^2(2t) + 4\cos^2(2t) + 1}, dt \\
=& \int_0^\pi \sqrt{5}, dt = \boxed{\sqrt{5}\pi}
\end{aligned}$$

(5) **Graph, name, and describe the set of all points in 3-space which satisfy the equation $z^2 - x^2 - y^2 = 1$.**

This graph is a hyperboloid of two sheets. Draw the hyperbola $z^2 - y^2 = 1$ in the yz-plane. Now rotate this hyperbola around the z-axis. We drew a picture on the picture page.

(6) **Find the absolute maximum and minimum values of**

$$f(x,y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

We drew the region on the picture page.

- We first look for interior points where f_x and f_y both vanish. We compute $f_x = 2 - 2x$ and $f_y = 4 - 2y$. We see that the only point where $f_x = 0$ and $f_y = 0$ is $\underline{(1,2)}$.

- We find points on $x = 0$ where the derivative of (f restricted to $x = 0$) vanishes. We are interested in $f(y) = 2 + 4y - y^2$ with $0 \leq y \leq 9$. We compute $f'(y) = 4 - 2y$. We consider $\underline{(0, 2)}$.
- We find points on $y = 9 - x$ where the derivative of (f restricted to $y = 9 - x$) vanishes. We are interested in

$$f(x) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2$$

  with $0 \leq x \leq 9$. We compute $f'(x) = 2 - 4 - 2x + 2(9 - x) = 16 - 4x$. We consider $\underline{(4, 5)}$.
- We find points on $y = 0$ where the derivative of (f restricted to $y = 0$) vanishes. We are interested in $f(x) = 2 + 2x - x^2$ for $0 \leq x \leq 9$. We compute $f'(x) = 2 - 2x$. We consider $\underline{(1, 0)}$.
- We also consider the corner points $\underline{(0, 0)}$, $\underline{(0, 9)}$, $\underline{(9, 0)}$.

The maximum and the minimum of f occur at one of the underlined points. We compute f at each of these points:

$$\begin{aligned}
f(1,2) &= 7 \\
f(0,2) &= 6 \\
f(4,5) &- 11 \\
f(1,0) &= 3 \\
f(0,0) &= 2 \\
f(0,9) &= -43 \\
f(9,0) &= -61
\end{aligned}$$

Thus,

The maximum of f on the given region is 7 and $f(1, 2) = 7$. The minimum of f on the given region is -61 and $f(9, 0) = -61$.

(7) **Find the volume of the solid between $z = 4 - x^2 - y^2$ and $z = x^2 + y^2 - 4$.**

Both equations represent paraboloids. The paraboloid on the left has vertex on the point $(0, 0, 4)$ and aims downwards. The paraboloid on the right has vertex on the point $(0, 0, -4)$ and aims upwards. The two surfaces meet when

$$4 - x^2 - y^2 = x^2 + y^2 - 4$$

$$8 = 2x^2 + 2y^2$$

$$4 = x^2 + y^2$$

We drew a picture on the picture page. We integrate in polar coordinates. We describe the fattest cross section by $0 \leq \theta \leq 2\pi$ and $0 \leq r \leq 4$.

For each point (r, θ) in the fattest cross section, z goes from $r^2 - 4$ to $4 - r^2$. The volume is equal to

$$\begin{aligned}
&\int_0^{2\pi} \int_0^2 \int_{r^2-4}^{4-r^2} r\,dz\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 2(4 - r^2) r\,dr\,d\theta \\
&= \int_0^{2\pi} \int_0^2 2(4r - r^3)\,dr\,d\theta \\
&= \int_0^{2\pi} 2(2r^2 - \tfrac{r^4}{4})\Big|_0^2\,d\theta \\
&= \int_0^{2\pi} 8 d\theta = \boxed{16\pi}
\end{aligned}$$

(8) **Compute $\int_0^1 \int_y^1 e^{x^2}\,dx\,dy$.**

None of know $\int e^{x^2}\,dx$. We have to do something clever. We try to change the order of integration. The present integral is computed over the region described by: For each fixed y with $0 \le y \le 1$, x goes from $x = y$ to $x = 1$. This fills up a triangle using horizontal lines.

On the picture pages we set up the same integral by filling up the region with vertical lines. The original integral is equal to

$$\begin{aligned}
&\int_0^1 \int_0^x e^{x^2}\,dy\,dx \\
&= \int_0^1 e^{x^2} y\Big|_0^x\,dx \\
&= \int_0^1 x e^{x^2}\,dx \\
&= \tfrac{1}{2} e^{x^2}\Big|_0^1 \\
&= \boxed{\tfrac{1}{2}(e - 1)}
\end{aligned}$$

(9) **Find the area of the region bounded by $x = -y^2$ and $y = x + 2$. (You must draw a meaningful picture.)**

The picture on the picture page shows that best way to fill the region is with horizontal lines. For each fixed y with $-2 \le y \le 1$, x goes from

$x = y - 2$ to $x = -y^2$. The area is equal to

$$
\begin{aligned}
&\int_{-2}^{1} \int_{y-2}^{-y^2} dx\, dy \\
&= \int_{-2}^{1} x\Big|_{y-2}^{-y^2} dy \\
&= \int_{-2}^{1} (-y^2 - (y-2))\, dy \\
&= \int_{-2}^{1} (-y^2 - y + 2)\, dy \\
&= \left(\tfrac{-y^3}{3} - \tfrac{y^2}{2} + 2y\right)\Big|_{-2}^{1} \\
&= -\tfrac{1}{3} - \tfrac{1}{2} + 2 - \left(\tfrac{8}{3} - \tfrac{4}{2} - 4\right) \\
&= -\tfrac{9}{3} - \tfrac{1}{2} + 2 + 2 + 4 \\
&= 5 - \tfrac{1}{2} \\
&= \boxed{\tfrac{9}{2}}
\end{aligned}
$$

(10) **Find parametric equations for the line tangent to the curve**

$$\vec{\boldsymbol{r}}(t) = t^2\vec{\boldsymbol{i}} + t^3\vec{\boldsymbol{j}}$$

at the point $(4,8)$**.**

The position vector $\vec{\boldsymbol{r}}(t)$ touches the point $(4,8)$ at time $t = 2$. We want the line which passes through $(4,8)$ and is parallel to $\vec{\boldsymbol{r}}'(2)$.

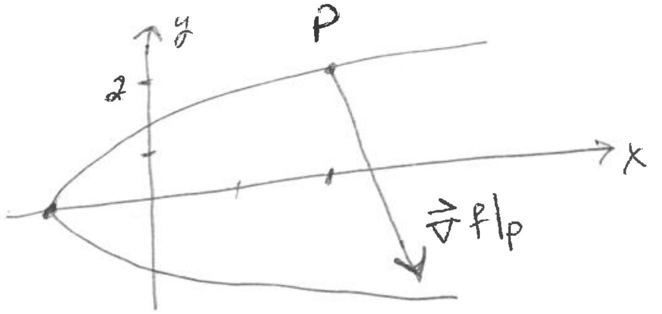
We compute $\vec{\boldsymbol{r}}'(t) = 2t\,\vec{\boldsymbol{i}} + 3t^2\vec{\boldsymbol{j}}$; hence, $\vec{\boldsymbol{r}}'(2) = 4\,\vec{\boldsymbol{i}} + 12\,\vec{\boldsymbol{j}}$. The line through $(4,8)$ parallel to $\vec{\boldsymbol{r}}'(2) = 4\,\vec{\boldsymbol{i}} + 12\,\vec{\boldsymbol{j}}$ is

$$\boxed{x = 4 + 4t, \quad y = 8 + 12t.}$$

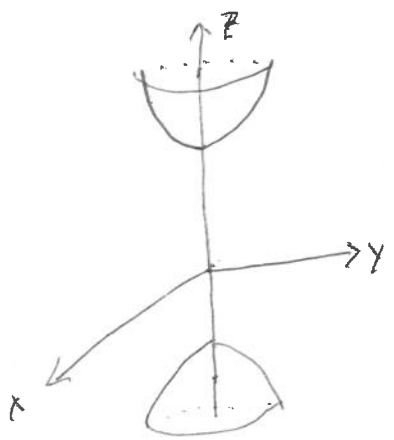
Of course, when one eliminates the parameter one gets $y = 3x - 4$, which is the equation of the line tangent to $y = x^{3/2}$ at $(4,8)$.

Final Exam Fall 2024 Picture Page

#3



#5



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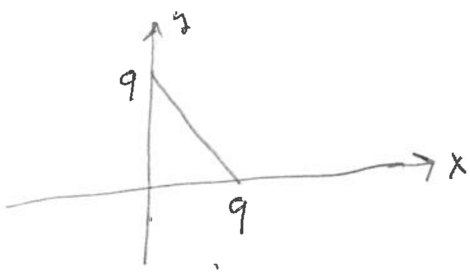

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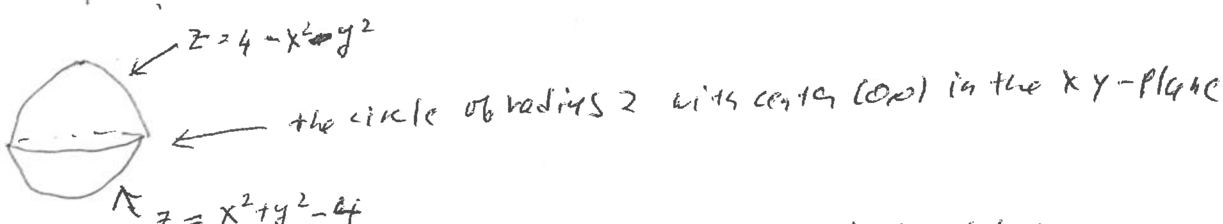
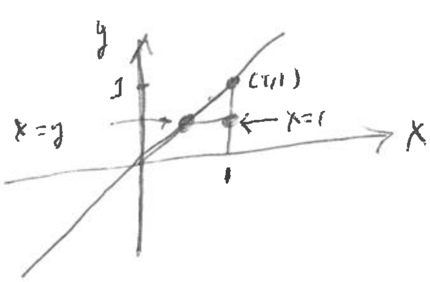


#8 The original integral is taken over a region defined by:
For each fixed y with $0 \le y \le 1$, x goes from $x = y$ to $x = 1$:



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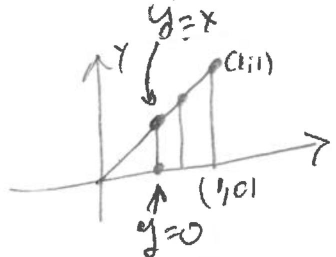
Picture Page 2

#8 continued

We fill up the same region using vertical lines



for each fixed x with $0 \le x \le 1$, y goes from $y=0$ to $y=x$

The integral becomes

$$\int_0^1 \int_0^x e^{x^2}\, dy\, dx$$

#9