Math 241, Exam 3, Fall, 2020

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, put the problems in order and send a picture of your solutions to

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The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Is there a plane that contains the lines**

$$\begin{cases} x = -t + 5 \\ y = 2t + 1 \\ z = -t - 1 \end{cases} \quad \textbf{and} \quad \begin{cases} x = -5t - 4 \\ y = -2t + 1 \\ z = 3t + 2? \end{cases}$$

If there is, then find its equation. Please make sure that your answer is correct.

The points $P_0 = (5, 1, -1)$ and $P_1 = (4, 3, -2)$ are on the first line. The point $Q = (-4, 1, 2)$ is on the second line. We find the plane which contains P_0, P_1, and Q. (It is NOT guaranteed ahead of time that there is a plane that contains the two lines; however, if such a plane exists, then this plane must contain P_0, P_1, and Q. It is absolutely required that we check that the plane we find does contain both lines.)

We calculate

$$\overrightarrow{P_0P_1} \times \overrightarrow{P_0Q} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -1 & 2 & -1 \\ -9 & 0 & 3 \end{vmatrix} = 6\,\vec{\boldsymbol{i}} + 12\,\vec{\boldsymbol{j}} + 18\,\vec{\boldsymbol{k}}.$$

The plane through $(5, 1, -1)$ perpendicular to $6\,\vec{\boldsymbol{i}} + 12\,\vec{\boldsymbol{j}} + 18\,\vec{\boldsymbol{k}}$ is

$$6(x-5) + 12(z-1) + 18(z+1) = 0.$$

Divide each side by 6 to obtain

$$(x-5) + 2(y-1) + 3(z+1) = 0$$

or

$$x + 2y + 3z = 4.$$

The line on the left lies on our plane because

$$(-t+5) + 2(2t+1) + 3(-t-1) = 4.\checkmark$$

The line the right lies on our plane because

$$(-5t-4)+2(-2t+1)+3(3t+2)=4.\checkmark$$

$$\boxed{\text{Both lines lie on the plane } x+2y+3z=4.}$$

(2) **An object moves in three space. At time t, the position vector of the object is $\vec{r}(t)=e^{2t}\vec{i}+(2t^2+3)\vec{j}+t^3\vec{k}$. What are parametric equations for the line tangent to the path of the object at $t=1$?**

We compute $\vec{r}'(t)=2e^{2t}\vec{i}+4t\vec{j}+3t^2\vec{k}$. The tangent line of interest is parallel to $\vec{r}'(1)=2e^2\vec{i}+4\vec{j}+3\vec{k}$. The position vector of the object at $t=1$ is $\vec{r}(1)=e^2\vec{i}+5\vec{j}+1\vec{k}$. The line through $(e^2,5,1)$ parallel to $2e^2\vec{i}+4\vec{j}+3\vec{k}$ is

$$\boxed{\begin{cases} x=2e^2t+e^2 \\ y=4t+5 \\ z=3t+1. \end{cases}}$$

(3) **An object moves in three space. At time t, the position vector of the object is $\vec{r}(t)=t\vec{i}+t^{3/2}\vec{j}$. How far does the object travel between $t=0$ and $t=4$?**

The length of the curve is

$$\int_0^4 |\vec{r}'(t)|dt=\int_0^4 |\vec{i}+(3/2)t^{1/2}\vec{j}|dt=\int_0^4 \sqrt{1^2+((3/2)t^{1/2})^2}dt$$

$$=\int_0^4 \sqrt{1+(9/4)t}dt=(2/3)(4/9)(1+(9/4)t)^{3/2}|_0^4=\boxed{(8/27)(10^{3/2}-1)}$$

(4) **Find the local maximum points, local minimum points, and saddle points of $f(x,y)=x^2y+4xy-2y^2$.**

The derivatives are

$$f_x=2xy+4y,\quad f_y=x^2+4x-4y,$$

$$f_{xx}=2y,\quad f_{xy}=2x+4,\quad \text{and}\quad f_{yy}=-4.$$

Observe that the equation $f_x=0$ can be factored to yield $2y(x+2)=0$. There are two different ways this equation can be satisfied: either $y=0$ or $x=-2$.

When $y=0$, then $f_y=0$ becomes $x^2+4x=0$; hence $x(x+4)=0$ and $x=0$ or $x=-4$. So far, we have identified two critical points; namely $(0,0)$ and $(-4,0)$.

When $x=-2$, then $f_y=0$ becomes $4-8-4y=0$; hence, $-4=4y$ and $y=-1$.

Thus f has exactly three critical points, namely $(0,0)$, $(-4,0)$, and $(-2,-1)$.

We apply the second derivative test at each critical point.

At $(0,0)$, the Hessian $H|_{(0,0)}$ is equal to

$$H|_{(0,0)} = (f_{xx}f_{yy}-f_{xy}^2)|_{(0,0)} = \Big((2y)(-4)-(2x+4)^2\Big)\Big|_{(0,0)} = -16<0.$$

Thus,

$$\boxed{(0,0,f(0,0)) \text{ is a saddle point.}}$$

At $(-4,0)$, the Hessian $H|_{(-4,0)}$ is equal to

$$H|_{(-4,0)} = (f_{xx}f_{yy}-f_{xy}^2)|_{(-4,0)} = \Big((2y)(-4)-(2x+4)^2\Big)\Big|_{(-4,0)} = -16<0.$$

Thus,

$$\boxed{(-4,0,f(-4,0)) \text{ is a saddle point.}}$$

At $(-2,-1)$, the Hessian $H|_{(-2,-1)}$ is equal to

$$H|_{(-2,-1)} = (f_{xx}f_{yy}-f_{xy}^2)|_{(-2,-1)} = \Big((2y)(-4)-(2x+4)^2\Big)\Big|_{(-2,-1)}$$
$$= \Big((-2)(-4)-(2(-2)+4)^2\Big)\Big|_{(-4,0)} = 8>0.$$

We see also that $f_{xx}(-2,-1)=2y|_{(-2,-1)}=-2<0$. We conclude that

$$\boxed{(-2,-1,f(-2,-1)) \text{ is a local maximum.}}$$

(5) **Find the absolute extreme points of the function $f(x,y)=x+y-xy$, which is defined on the closed triangle with vertices at $(0,0)$, $(0,2)$, and $(4,0)$.**

We put a picture of the domain on the last page. We see that the boundary has three pieces. Eventually, we will look at f restricted to each of these three pieces. Eventually, also, we will look at f evaluated at each of the end points of the boundary.

First we look for interior points where both partial derivatives vanish. We compute $f_x=1-y$ and $f_y=1-x$. If $f_x=0$ and $f_y=0$ then $x=1$ and $y=1$. We will study $(1,1)$ in our final step.

Now we look at f restricted to the vertical line $x=0$, with $0\le y\le 2$. This function is

$$f|_{x=0}=y.$$

We see that $\frac{d}{dy}(f|_{x=0}) = 1$, which is never zero. Thus, the extreme points of $f|_{x=0}$ occur at the end points $(0, 0)$ and $(0, 2)$. We already know to study these points in our final step.

Now we look at f restricted to the horizontal line $y = 0$, with $0 \leq x \leq 4$. This function is

$$f|_{y=0} = x.$$

We see that $\frac{d}{dx}(f|_{y=0}) = 1$, which is never zero. Thus, the extreme points of $f|_{y=0}$ occur at the end points $(0, 0)$ and $(4, 0)$. We already know to study these points in our final step.

Now we look at f restricted to the slanting line $y = -\frac{1}{2}x + 2$, with $0 \leq x \leq 4$. This function is

$$f|_{y=-\frac{1}{2}x+2} = x + (-\tfrac{1}{2}x + 2) - x(-\tfrac{1}{2}x + 2) = \tfrac{x^2}{2} - \tfrac{3}{2}x + 2.$$

We compute

$$\tfrac{d}{dx}(f|_{\text{slanting line}}) = x - \tfrac{3}{2}.$$

Thus, $\frac{d}{dx}(f|_{\text{slanting line}}) = 0$ when $x = \frac{3}{2}$ and $y = 2 - \frac{3}{4} = \frac{5}{4}$.

It is time for the final step. The extreme points of f on our domain occur at one of the points $(0, 0)$, $(0, 2)$, $(4, 0)$, $(1, 1)$, or $(\frac{3}{2}, \frac{5}{4})$. We evaluate f at these 5 points; the largest answer is the maximum. The smallest answer is the minimum.

$$\begin{aligned} f(0,0) &= 0 \\ f(0,2) &= 2 \\ f(4,0) &= 4 \\ f(1,1) &= 1 \\ f(\tfrac{3}{2}, \tfrac{5}{4}) &= \tfrac{7}{8} \end{aligned}$$

We conclude that $(4, 0, 4)$ is the maximum of f on our domain and $(0, 0, 0)$ is the minimum of f on our domain.

Picture for number 5