Math 241, Exam 3, Spring, 2022

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points: problems one to four are worth 8 points each; problems five and six are worth 9 points each.

Make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the point on the plane $x + 2y + 3z = 4$ which is closest to the point $(1, 4, 6)$.**

Let ℓ be the line through $(1, 4, 6)$ which is perpendicular to the plane $x + 2y + 3z = 4$. Thus, ℓ is

$$\begin{cases} x = 1 + t \\ y = 4 + 2t \\ z = 6 + 3t \end{cases}$$

The line ℓ intersects the plane $x + 2y + 3z = 4$ when

$$(1 + t) + 2(4 + 2t) + 3(6 + 3t) = 4.$$

The line intersects the plane when $t = -23/14$. The point of intersection, which is $\boxed{\left(\frac{-9}{14}, \frac{10}{14}, \frac{15}{14}\right)}$, is the point on $x + 2y + 3z = 4$ which is nearest to $(1, 4, 6)$.

(2) **An object moves on the xy-plane. The position vector of the object at time t is $\overrightarrow{\boldsymbol{r}}(t) = t^2\,\overrightarrow{\boldsymbol{i}} + t^3\,\overrightarrow{\boldsymbol{j}}$. How far does the object travel between $t = 0$ and $t = 1$?**

The distance traveled is equal to

$$\int_0^1 |\overrightarrow{\boldsymbol{r}}'(t)|dt = \int_0^1 |2t\,\overrightarrow{\boldsymbol{i}} + 3t^2\,\overrightarrow{\boldsymbol{j}}|dt = \int_0^1 \sqrt{4t^2 + 9t^4}dt = \int_0^1 t\sqrt{4 + 9t^2}dt$$

$$= \frac{2}{3}\frac{(4 + 9t^2)^{3/2}}{18}\Big|_0^1 = \boxed{\frac{13\sqrt{13}}{27} - \frac{8}{27}}.$$

(3) **Find the directional derivative of the function $f(x,y) = x^2 + 3y^2$ in the direction of $\vec{\boldsymbol{u}} = 2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}$ at the point $P = (3,4)$.**

We compute

$$D_{\vec{\boldsymbol{u}}} f|_P = (\vec{\nabla} f)|_P \cdot \frac{\vec{\boldsymbol{u}}}{|\vec{\boldsymbol{u}}|} = (2x\vec{\boldsymbol{i}} + 6y\vec{\boldsymbol{j}})|_{(3,4)} \cdot \frac{(2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}})}{\sqrt{13}}$$
$$= \frac{1}{\sqrt{13}}(6\vec{\boldsymbol{i}} + 24\vec{\boldsymbol{j}}) \cdot (2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}) = \boxed{\frac{1}{\sqrt{13}}(12+72)}.$$

(4) **Find the equation of the plane tangent to $z = 3x^2 + y^2$ at the point where $x = 1$ and $y = 2$.**

Gradients are perpendicular to level sets. We view $z = 3x^2 + y^2$ as the level set $g(x,y,z) = 0$, where $g(x,y,z) = 3x^2 + y^2 - z$. The answer is the plane through $(1,2,7)$ perpendicular to

$$\vec{\nabla} g|_{(1,2,7)} = (6x\vec{\boldsymbol{i}} + 2y\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}})|_{1,2,7} = 6\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}.$$

The tangent plane is

$$\boxed{6(x-1) + 4(y-2) - (z-7) = 0.}$$

(5) **Find all local maximum points, local minimum points, and saddle points of $f(x,y) = 4 + x^3 + y^3 - 3xy$.**

We compute $f_x = 3x^2 - 3y$ and $f_y = 3y^2 - 3x$. Both partial derivatives are zero when

$$3x^2 - 3y = 0 \quad \text{and} \quad 3y^2 - 3x = 0.$$

Both partial derivatives are zero when

$$x^2 = y \quad \text{and} \quad 3(x^2)^2 - 3x = 0.$$

Both partial derivatives are zero when

$$x^2 = y \quad \text{and} \quad x(x^3 - 1) = 0.$$

Both partial derivatives are zero when $(x,y) = (0,0)$ or $(x,y) = (1,1)$. We apply the second derivative test at each of these critical points.

We compute $f_{xx} = 6x$, $f_{xy} = -3$, and $f_{yy} = 6y$. It follows that

$$H(x,y) = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9.$$

We see that $H(0,0) = -9 < 0$. We conclude that

$$\boxed{(0,0,4) \text{ is a saddle point of } z = f(x,y).}$$

We see that $H(1,1) = 36 - 9 > 0$ and $f_{xx}(1,1) = 6 > 0$. We conclude that

$$\boxed{(1,1,3) \text{ is a local minimum point of } z = f(x,y).}$$

(6) **Find the absolute maximum and minimum values of the function** $f(x,y) = -x^2 - y^2 + 2x + 2y + 1$ **on the triangular region in the first quadrant bounded by the lines** $x = 0$**,** $y = 0$**, and** $y = 2 - x$**.**

The vertices of the region, namely $\underline{(0,0)}$, $\underline{(0,2)}$, and $\underline{(2,0)}$, all are points on interest.

We find all interior points where both partial derivatives vanish. We compute $f_x = -2x + 2$ and $f_y = -2y + 2$. Thus $f_x = 0$ and $f_y = 0$ at $\underline{(1,1)}$, which is another point of interest.

The restriction of f to $x = 0$ is $f|_{x=0} = -y^2 + 2y + 1$. We compute $\frac{d}{dy}(f|_{x=0}) = -2y + 2$. This derivative is zero at $(x,y) = \underline{(0,1)}$, which is a point of interest.

The restriction of f to $y = 0$ is $f|_{y=0} = -x^2 + 2x + 1$. We compute $\frac{d}{dx}(f|_{y=0}) = -2x + 2$. This derivative is zero at $(x,y) = \underline{(1,0)}$, which is a point of interest.

The restriction of f to $y = 2 - x$ is

$$f|_{y=2-x} = -x^2 - (2-x)^2 + 2x + 2(2-x) + 1 = -2x^2 + 4x + 1.$$

We compute $\frac{d}{dx}(f|_{y=2-x}) = -4x + 4$. This derivative is zero at $(x,y) = \underline{(1,1)}$, which is a point of interest. We compute

$$\begin{aligned} f(0,0) &= 1 \\ f(0,2) &= 1 \\ f(2,0) &= 1 \\ f(1,1) &= 3 \\ f(0,1) &= 2 \\ f(1,0) &= 2 \end{aligned}$$

The maximum of f on the domain occurs at $(1,1,3)$. The minimum of f on the domain occurs at $(0,0,1)$, $(0,2,1)$, and $(2,0,1)$.