Math 241, Exam 3, Spring, 2019

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Thursday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Describe and graph** $y^2 - x^2 - z^2 = 1$ **in three-space. What is the name of this object?**

See the picture page.

(2) **Find the length of the curve** $y = \frac{2}{3}x^{3/2}$ **for** $0 \le x \le 8$**.**

The curve is parameterized by $\vec{\boldsymbol{r}}(t) = t\,\vec{\boldsymbol{i}} + \frac{2}{3}t^{3/2}$ for $0 \le t \le 8$. The length of the curve is

$$\int_0^8 |\vec{\boldsymbol{r}}\,'(t)|dt = \int_0^8 |\vec{\boldsymbol{i}} + t^{1/2}\vec{\boldsymbol{j}}|dt = \int_0^8 \sqrt{1+t}dt = \frac{2}{3}(1+t)^{3/2}\Big|_0^8 = \boxed{\frac{2}{3}(27-1)}.$$

(3) **Find the equation of the plane tangent to** $z = x^2 + y^2$ **at the point where** $x = 3$ **and** $y = 4$**.**

When $x = 3$ and $y = 4$, then $z = 25$. Gradients are perpendicular to level sets. Move the z to the other side. We want the equation of the plane through the point $(3, 4, 25)$ and perpendicular to the vector

$$\vec{\boldsymbol{\nabla}}(x^2 + y^2 - z)|_{(3,4,25)} = (2x\,\vec{\boldsymbol{i}} + 2y\,\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}})|_{(3,4,25)} = 6\,\vec{\boldsymbol{i}} + 8\,\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}.$$

The answer is

$$\boxed{6(x-3) + 8(y-4) - (z-25) = 0.}$$

(4) **Find the absolute maximum and absolute minimum of**

$$f(x, y) = 2 + 2x + 4y - x^2 - y^2$$

on the triangular region in the first quadrant bounded by the lines $x = 0$, $y = 0$, and $y = 9 - x$.

See the picture page for a picture of the domain.

We find all points in the interior where both partials vanish. Observe that $f_x = 2 - 2x$ and $f_y = 4 - 2y$. Both partials vanish at $\underline{(1, 2)}$. This point is in the interior.

We find all points on $x = 0$ with $0 \le y \le 9$ where the derivative of $f(0, y) = 2 + 4y - y^2$ is zero. The derivative is $4 - 2y$. This is zero when $y = 2$. We must study $\underline{(0, 2)}$.

We find all points on $y = 0$ with $0 \le x \le 9$ where the derivative of $f(x, 0) = 2 + 2x - x^2$ is zero. The derivative is $2 - 2x$. The derivative is zero when $x = 1$. We must study $\underline{(1, 0)}$.

We find all points on $y = 9 - x$ with $0 \le x \le 9$ where the derivative of $f(x, 9 - x) = 2 + 2x + 4(9 - x) - x^2 - (9 - x)^2$ is zero. The derivative is $2 - 4 - 2x + 2(9 - x) = 16 - 4x$. The derivative is zero when $x = 4$. We must study $\underline{(4, 5)}$.

We must also study the end points of the boundary; namely, $\underline{(0, 0)}$, $\underline{(0, 9)}$, and $\underline{(9, 0)}$.

We plug all 7 candidates into f:

$$
\begin{aligned}
f(1, 2) &= 2 + 2 + 8 - 1 - 4 = 7 \\
f(0, 2) &= 2 + 8 - 4 = 6 \\
f(1, 0) &= 2 + 2 - 1 = 3 \\
f(4, 5) &= 2 + 8 + 20 - 16 - 25 = -11 \\
f(0, 0) &= 2 \\
f(0, 9) &= 2 + 36 - 81 = -43 \\
f(9, 0) &= 2 + 18 - 81 = -61
\end{aligned}
$$

The maximum point of f on the given domain is $(1, 2, 7)$. The minimum point of f on the given domain is $(9, 0, -61)$.

(5) **Find the volume of the solid whose base in the xy-plane is the region bounded by $x + y = 2$ and $x + 4 = y^2$ and whose top is $z = x + 5$.**

Find the intersection of $x + y = 2$ and $x + 4 = y^2$:

$$2 - y + 4 = y^2$$

$$0 = y^2 + y - 6$$

$$0 = (y + 3)(y - 2)$$

$$y = -3 \quad \text{or} \quad y = 2$$

The intersection points are $(5, -3)$ and $(0, 2)$.

Look at the picture page. We fill up the base using horizontal lines. Fix y with $-3 \le y \le 2$. For each fixed y, x goes from $y^2 - 4$ to $2 - y$. The volume is

$$\int\int_{\text{base}} \text{top } dA = \int_{-3}^{2}\int_{y^2-4}^{2-y} (x+5)dx\,dy = \int_{-3}^{2} (x^2/2 + 5x)\Big|_{y^2-4}^{2-y} dy$$

$$= \int_{-3}^{2} \frac{(2-y)^2}{2} + 5(2-y) - \Big(\frac{(y^2-4)^2}{2} + 5(y^2-4)\Big) dy$$

$$= \int_{-3}^{2} \frac{4-4y+y^2}{2} + 5(2-y) - \Big(\frac{y^4-8y^2+16}{2} + 5(y^2-4)\Big) dy$$

$$= \Big[\frac{4y-2y^2+y^3/3}{2} + 5(2y-y^2/2) - \Big(\frac{y^5/5-8y^3/3+16y}{2} + 5(y^3/3-4y)\Big)\Big]\Big|_{-3}^{2}$$

$$= \begin{cases} \Big[\frac{4(2)-2(2)^2+(2)^3/3}{2} + 5(2(2)-(2)^2/2) - \Big(\frac{(2)^5/5-8(2)^3/3+16(2)}{2} + 5((2)^3/3-4(2))\Big)\Big] \\ -\Big[\frac{4(-3)-2(-3)^2+(-3)^3/3}{2} + 5(2(-3)-(-3)^2/2) - \Big(\frac{(-3)^5/5-8(-3)^3/3+16(-3)}{2} + 5((-3)^3/3-4(-3))\Big)\Big] \end{cases}$$