Math 241, Exam 3, Fall, 2018

Write everything on the blank paper provided. **YOU SHOULD KEEP THIS PIECE OF PAPER.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Tuesday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find all local minima, local maxima and saddle points for the function $f(x,y) = x^2 + 4y^2 - 6x + 8y - 15$.**

We compute $\frac{\partial f}{\partial x} = 2x - 6$ and $\frac{\partial f}{\partial y} = 8y + 8$. Both partial derivatives are zero at the point $(x,y) = (3,-1)$. We apply the second derivative test at that point. We compute $\frac{\partial^2 f}{\partial x^2} = 2$, $\frac{\partial^2 f}{\partial y \partial x} = 0$, and $\frac{\partial^2 f}{\partial y^2} = 8$.

Observe that

$$\left(\frac{\partial^2 f}{\partial x^2}\frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial y \partial x}\right)^2 \right)\Bigg|_{(x,y)=(3,-1)} = 16,$$

which is positive. Thus, $(3,-1,f(3,1))$ is not a saddle point; it is either a local maximum or a local minimum. Also

$$\frac{\partial^2 f}{\partial x^2}\Bigg|_{(x,y)=(3,-1)} = 2,$$

which is positive. We conclude that

$$\boxed{(3,-1,f(3,1)) \text{ is a local minimum.}}$$

(2) **Find the absolute maximum and the absolute minimum values of**

$$f(x,y) = 3xy - 6x - 3y + 7$$

on the triangular region with vertices $(0,0)$, $(3,0)$, and $(0,5)$.

We compute $\frac{\partial f}{\partial x} = 3y - 6$ and $\frac{\partial f}{\partial y} = 3x - 3$. Both partial derivatives are zero at the point $(x,y) = (1,2)$. This interior critical point might be an absolute extreme point.

We look at the restriction of f to the line $y = 0$ with $0 \le x \le 3$.

$$f|_{y=0}(x) = -6x + 7$$

and the derivative is -6 which is never zero. So the absolute extreme points of the restriction of f to the line $y = 0$ with $0 \le x \le 3$ occur at the end points.

We look at the restriction of f to the line $x = 0$ with $0 \le y \le 5$.

$$f|_{x=0}(y) = -3y + 7$$

and the derivative is -3 which is never zero. So the absolute extreme points of the restriction of f to the line $x = 0$ with $0 \le y \le 5$ occur at the end points.

The line which connects $(3,0)$ to $(0,5)$ is $y = -\frac{5}{3}x + 5$. We look at the restriction of f to the line $y = -\frac{5}{3}x + 5$ with $0 \le x \le 3$.

$$f|_{\text{top boundary}}(x) = 3x(-\frac{5}{3}x+5) - 6x - 3(-\frac{5}{3}x+5) + 7 = -5x^2 + 15x - 6x + 5x - 15 + 7$$

The derivative is $-10x + 15 - 6 + 5$. The derivative is zero when $-10x + 14 = 0$; that is when $x = 7/5$ and $y = 8/3$.

The extreme points of f on our domain occur at $(0,0)$, $(3,0)$, $(0,5)$, or $(7/5, 8/3)$. We plug these points into f:

$$\begin{aligned} f(0,0) &= 7, && \text{absolute maximum} \\ f(3,0) &= -18 + 7 = -11 && \text{absolute minimum} \\ f(0,5) &= -15 + 7 = -8 \\ f(7/5, 8/3) &= 56/5 - 42/5 - 8 + 7 = 9/5 \\ f(1,2) &= 6 - 6 - 6 + 7 = 1 \end{aligned}$$

The absolute minimum of f on our domain occurs at $(3, 0, -11)$.
The absolute maximum of f on our domain occurs at $(0, 0, 7)$.

(3) **Describe, graph, and name** $x^2 + y^2 - z^2 = 1$ **in 3-space.**

See the picture page.

(4) **Suppose** $\vec{r}\,'(t) = 2t\,\vec{i} + 3t^2\,\vec{j}$ **and** $\vec{r}(0) = \vec{i} - \vec{j}$**. Find** $\vec{r}(t)$**.**

Integrate to see that $\vec{r}(t) = t^2\,\vec{i} + t^3\,\vec{j} + \vec{c}$ for some constant vector $\vec{c}$. Plug in $t = 0$ to see that $\vec{i} - \vec{j} = \vec{r}(0) = \vec{c}$. Thus

$$\boxed{\vec{r}(t) = (t^2 + 1)\,\vec{i} + (t^3 - 1)\,\vec{j}.}$$

(5) **Let $f(x, y) = 4x^3y^2$. Find the directional derivative of f at the point $P = (2, 1)$ in the direction of $\vec{\boldsymbol{a}} = 4\vec{\boldsymbol{i}} - 3\vec{\boldsymbol{j}}$.**

The unit vector which points in the direction of $\vec{\boldsymbol{a}}$ is $vu = \frac{1}{5}(4\vec{\boldsymbol{i}} - 3\vec{\boldsymbol{j}})$.

$$D_{\vec{\boldsymbol{u}}} f|_P = (\vec{\nabla} f)|_P \cdot \vec{\boldsymbol{u}} = (12x^2y^2\vec{\boldsymbol{i}} + 8x^3y\vec{\boldsymbol{j}})|_{(2,1)} \cdot \frac{1}{5}(4\vec{\boldsymbol{i}} - 3\vec{\boldsymbol{j}})$$

$$= (48\vec{\boldsymbol{i}} + 64\vec{\boldsymbol{j}}) \cdot \frac{1}{5}(4\vec{\boldsymbol{i}} - 3\vec{\boldsymbol{j}}) = \frac{1}{5}(4(48) - 3(64)) = \boxed{0}.$$