Math 241, Exam 2, Fall, 2020

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

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You should KEEP this piece of paper. If possible: put the problems in order before you take your picture. (Use as much paper as necessary).

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Find the point on the plane $x+2y+3z=4$ which is closest to the point $(2,3,4)$.**

The vector $\overrightarrow{\boldsymbol{N}} = \overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}} + 3\overrightarrow{\boldsymbol{k}}$ is perpendicular to the plane. The line

$$\begin{cases} x = 2+t \\ y = 3+2t \\ z = 4+3t \end{cases}$$

passes through the point $(2,3,4)$ and is perpendicular to the plane. The line and the plane intersect when

$$(2+t)+2(3+2t)+3(4+3t)=4.$$

The line and the plane intersect at $t=-\frac{8}{7}$. The point of intersection is $\boxed{(\frac{6}{7},\frac{5}{7},\frac{4}{7})}$. Of course, this point of intersection is the point on the plane which is closest to $(2,3,4)$.

(2) **Graph, describe, and name the surface $x^2+y^2=z^2$ in three space.**

When $x=0$, the graph is $y^2=z^2$, which is two lines. When $y=0$, the graph is $x^2=z^2$, which is two lines. When $z=0$, the graph is a single point. When z is any non-zero constant, the graph is a circle.

The graph of $x^2+y^2=z^2$ in three space is a cone. See the last page for a picture.

(3) **An object moves in three space. At time t, the position vector of the object is $\overrightarrow{\boldsymbol{r}}(t)$. Suppose $\overrightarrow{\boldsymbol{r}}''(t) = \sin(2t)\overrightarrow{\boldsymbol{i}} + e^{2t}\overrightarrow{\boldsymbol{j}}$, $\overrightarrow{\boldsymbol{r}}(0) = \overrightarrow{\boldsymbol{i}} + 2\overrightarrow{\boldsymbol{j}}$, and $\overrightarrow{\boldsymbol{r}}'(0) = 3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}}$. Find $\overrightarrow{\boldsymbol{r}}(t)$.**

Integrate to see that

$$\vec{\boldsymbol{r}}'(t) = -\tfrac{1}{2}\cos(2t)\,\vec{\boldsymbol{i}} + \tfrac{1}{2}e^{2t}\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1$$

for some constant vector $\vec{\boldsymbol{c}}_1$. Plug in $t = 0$ to learn that

$$3\,\vec{\boldsymbol{i}} + 4\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{r}}'(0) = -\tfrac{1}{2}\cos(0)\,\vec{\boldsymbol{i}} + \tfrac{1}{2}e^{0}\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1 = -\tfrac{1}{2}\,\vec{\boldsymbol{i}} + \tfrac{1}{2}\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1.$$

Thus,

$$\tfrac{7}{2}\,\vec{\boldsymbol{i}} + \tfrac{7}{2}\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{c}}_1$$

and

$$\vec{\boldsymbol{r}}'(t) = (-\tfrac{1}{2}\cos(2t) + \tfrac{7}{2})\,\vec{\boldsymbol{i}} + (\tfrac{1}{2}e^{2t} + \tfrac{7}{2})\,\vec{\boldsymbol{j}}.$$

Integrate again to see that

$$\vec{\boldsymbol{r}}(t) = (-\tfrac{1}{4}\sin(2t) + (\tfrac{7}{2})t)\,\vec{\boldsymbol{i}} + (\tfrac{1}{4}e^{2t} + (\tfrac{7}{2})t)\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_2$$

for some constant vector $\vec{\boldsymbol{c}}_2$. Plug in $t = 0$ to learn that

$$\vec{\boldsymbol{i}} + 2\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{r}}(0) = (-\tfrac{1}{4}\sin(0))\,\vec{\boldsymbol{i}} + (\tfrac{1}{4}e^{0})\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_2 = (\tfrac{1}{4})\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_2.$$

Thus,

$$\vec{\boldsymbol{i}} + \tfrac{7}{4}\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{c}}_2$$

and

$$\boxed{\vec{\boldsymbol{r}}(t) = (-\tfrac{1}{4}\sin(2t) + (\tfrac{7}{2})t + 1)\,\vec{\boldsymbol{i}} + (\tfrac{1}{4}e^{2t} + (\tfrac{7}{2})t + \tfrac{7}{4})\,\vec{\boldsymbol{j}}}.$$

(4) **Suppose $z = \sqrt{x\sin(xy)}$. Find $\frac{\partial z}{\partial x}$ and $\frac{\partial z}{\partial y}$.**

We compute

$$\boxed{\frac{\partial z}{\partial x} = \frac{xy\cos(xy) + \sin(xy)}{2\sqrt{x\sin(xy)}} \quad \text{and} \quad \frac{\partial z}{\partial y} = \frac{x^2\cos(xy)}{2\sqrt{x\sin(xy)}}}.$$

(5) **Find the equation of the plane which contains the points $P_1 = (6, 3, -1)$, $P_2 = (1, -4, 1)$, and $P_3 = (2, 3, 4)$. Check your answer. Make sure it is correct.**

The vector

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -5 & -7 & 2 \\ -4 & 0 & 5 \end{vmatrix} = \begin{vmatrix} -7 & 2 \\ 0 & 5 \end{vmatrix}\vec{\boldsymbol{i}} - \begin{vmatrix} -5 & 2 \\ -4 & 5 \end{vmatrix}\vec{\boldsymbol{j}} + \begin{vmatrix} -5 & -7 \\ -4 & 0 \end{vmatrix}\vec{\boldsymbol{k}}$$

$$= -35\,\vec{\boldsymbol{i}} + 17\,\vec{\boldsymbol{j}} - 28\,\vec{\boldsymbol{k}}$$

is perpendicular to our plane. The plane through $(6, 3, -1)$ perpendicular to $-35\vec{\boldsymbol{i}} + 17\vec{\boldsymbol{j}} - 28\vec{\boldsymbol{k}}$ is

$$-35(x-6) + 17(y-3) - 28(z+1) = 0$$

$$\boxed{-35x + 17y - 28z = -131.}$$

Make sure that $P_1 = (6, 3, -1)$, $P_2 = (1, -4, 1)$, and $P_3 = (2, 3, 4)$ all satisfy your answer:

$$-35(6) + 17(3) + 28 = -131$$

$$-35(1) + 17(-4) - 28(1) = -131$$

$$-35(2) + 17(3) - 28(4) = -131.$$

$x^2 + y^2 = z^2$ is a cone