Math 241, Exam 2, Spring, 2021

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

kustin@math.sc.edu

ALSO, LEAVE A PHYSICAL COPY OF YOUR SOLUTIONS WITH ME. Fold your solutions in half and write your name on the outside.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Write** $2x^2+3y^2-12x+24y+47=0$ **in the form** $a(x-x_0)^2+b(y-y_0)^2=c$**, for some numbers** a**,** b**,** c**,** x_0**, and** y_0**.**

The original equation is the same as

$$2(x^2-6x+\boxed{})+3(y^2+8y+\boxed{\boxed{}})=-47+2\boxed{}+3\boxed{\boxed{}}.$$

Put $(\frac{-6}{2})^2=9$ into $\boxed{}$ and $(\frac{8}{2})^2=16$ into $\boxed{\boxed{}}$ in order to obtain

$$2(x^2-6x+\boxed{9})+3(y^2+8y+\boxed{\boxed{16}})=-47+2\boxed{9}+3\boxed{\boxed{16}}.$$

This is the same as

$$\boxed{2(x-3)^2+3(y+4)^2=19}.$$

(2) **Consider the function** $f(x,y)=\sqrt{x^2+y^2-1}$**.**

(a) **Graph and label a few level sets of the form** $f(x,y)=c$**, where** c **is a constant.**

(b) **Graph** $z=f(x,y)$**.**

We attack (a). The expression $\sqrt{x^2+y^2-1}$ is always zero or higher. So the level set $c=f(x,y)$ only makes sense for non-negative numbers c. If c is a non-negative constant, then $c=f(x,y)$ is the same as $c=\sqrt{x^2+y^2-1}$, which is the same as $c^2+1=x^2+y^2$, which is the circle with center $(0,0)$ and radius $\sqrt{c^2+1}$. In particular,

- when $c=0$, the corresponding level set is the circle with center $(0,0)$ and radius 1;

- when $c = \sqrt{3}$, the corresponding level set is the circle with center $(0,0)$ and radius 2;
- when $c = \sqrt{8}$, the corresponding level set is the circle with center $(0,0)$ and radius 3;
- when $c = \sqrt{15}$, the corresponding level set is the circle with center $(0,0)$ and radius 4;
- etc.

A picture of these level sets appears on the next page.

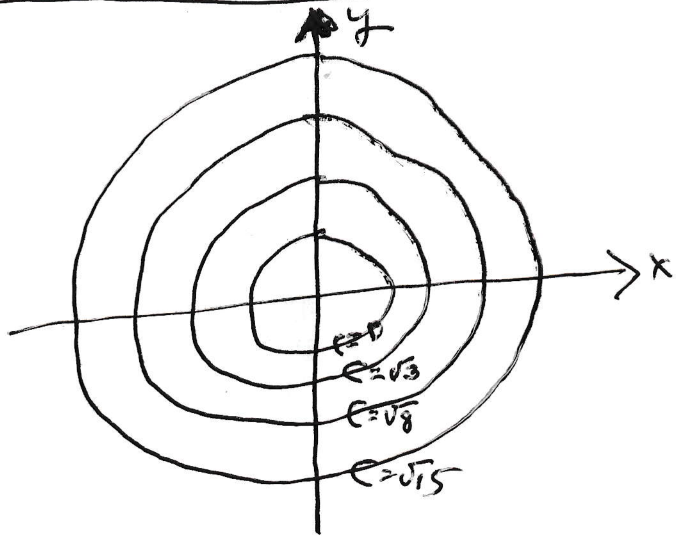
(b) The graph of (b) is the surface $z = \sqrt{x^2 + y^2 - 1}$ in 3-space. Possibly it is helpful to square both sides and observe that the answer to (b) is the PART of $z^2 = x^2 + y^2 - 1$ where z is positive. We already studied what happens when z is held constant. If $x = 0$, then the equation becomes $1 = y^2 - z^2$ and the graph is a hyperbola in the yz-plane which includes the point $(y, z) = (1, 0)$. If $y = 0$, then then the equation becomes $1 = x^2 - z^2$ and the graph is a hyperbola in the xz-plane which includes the point $(x, z) = (1, 0)$.

At any rate $z^2 = x^2 + y^2 - 1$ is the hyperboloid of one sheet with the z-axis sitting in the hollow part and $z = \sqrt{x^2 + y^2 - 1}$ is the part of the hyperboloid where z is positive.

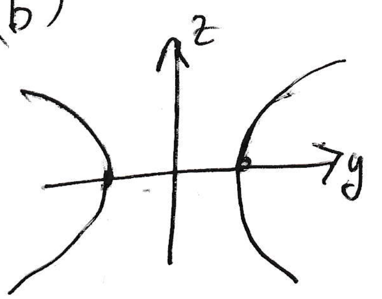
Pictures appear on the next page.

Pictures for number 2

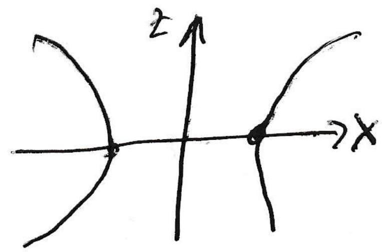
(a)



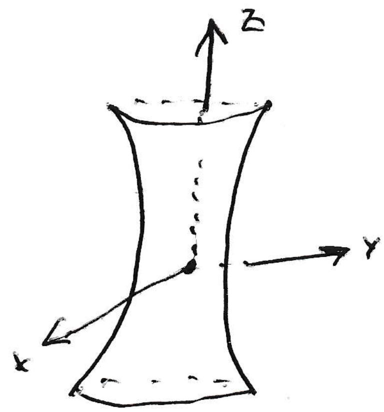
(b)



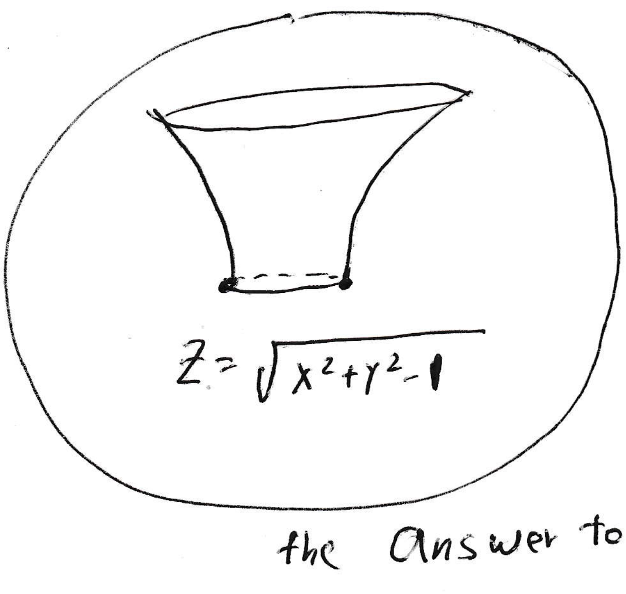
$1 = y^2 - z^2$



$1 = x^2 - z^2$



$z^2 = x^2 + y^2 - 1$



the answer to b

(3) **Let $f(x,y) = 2x^2y^3\sin(x^4y^5) + e^{2x} + \ln(5y)$. Find $\frac{\partial f}{dx}$.**

$$\boxed{\frac{\partial f}{dx} = \big(2x^2y^3\cos(x^4y^5)\big)4x^3y^5 + 4xy^3\sin(x^4y^5) + 2e^{2x}}$$

(4) **Find the equation of the plane which contains the points $P = (1,2,3)$, $Q = (4,5,6)$, and $R = (-1,0,2)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{PQ} \times \overrightarrow{PR}$ is perpendicular to the plane. We compute

$$\overrightarrow{PQ}\times\overrightarrow{PR} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 3 & 3 & 3 \\ -2 & -2 & -1 \end{vmatrix} = \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix}\vec{\boldsymbol{i}} - \begin{vmatrix} 3 & 3 \\ -2 & -1 \end{vmatrix}\vec{\boldsymbol{j}} + \begin{vmatrix} 3 & 3 \\ -2 & -2 \end{vmatrix}\vec{\boldsymbol{k}} = 3\,\vec{\boldsymbol{i}} - 3\,\vec{\boldsymbol{j}}.$$

The plane through $(-1,0,2)$ perpendicular to $3\,\vec{\boldsymbol{i}} - 3\,\vec{\boldsymbol{j}}$ is

$$3(x+1) - 3(y-0) + 0(z-2) = 0.$$

We can divide the equation by 3 and rewrite the equation as

$$x - y + 1 = 0.$$

$$\boxed{\text{The plane which contains } P,\ Q, \text{ and } R \text{ is } x - y + 1.}$$

Check: The point P satisfies the proposed answer because $1-2+1=0$. The point Q satisfies the proposed answer because $4-5+1=0$. The point R satisfies the proposed answer because $-1-0+1=0$.

(5) **An object travels in the xy-plane. The position vector of the object at time t is $\vec{\boldsymbol{r}}(t)$, for $0 \le t$. The acceleration vector of the object at time t is $\vec{\boldsymbol{r}}''(t) = 6\,\vec{\boldsymbol{i}} + 16e^{2t}\,\vec{\boldsymbol{j}}$. The initial position vector of the object is $\vec{\boldsymbol{r}}(0) = 6\,\vec{\boldsymbol{j}}$ and the initial velocity vector of the object is $\vec{\boldsymbol{r}}'(0) = 9\,\vec{\boldsymbol{j}}$. Find the y-coordinate of the object when the x-coordinate is 3.**

We integrate $\vec{\boldsymbol{r}}''(t)$ to learn that $\vec{\boldsymbol{r}}'(t) = 6t\,\vec{\boldsymbol{i}} + 8e^{2t}\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1$ for some constant vector $\vec{\boldsymbol{c}}_1$. Plug in $t=0$ to see that

$$9\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{r}}'(0) = 8\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1.$$

It follows that $\vec{\boldsymbol{c}}_1 = \vec{\boldsymbol{j}}$ and

$$\vec{\boldsymbol{r}}'(t) = 6t\,\vec{\boldsymbol{i}} + (8e^{2t}+1)\,\vec{\boldsymbol{j}}.$$

Integrate $\vec{\boldsymbol{r}}'(t)$ to learn that

$$\vec{\boldsymbol{r}}(t) = 3t^2\,\vec{\boldsymbol{i}} + (4e^{2t}+t)\,\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_2$$

for some constant vector $\vec{c}_2$. Plug in $t = 0$ to see that

$$6\vec{j} = \vec{r}(0) = 4\vec{j} + \vec{c}_2.$$

It follows that $\vec{c}_2 = 2\vec{j}$ and

$$\vec{r}(t) = 3t^2\vec{i} + (4e^{2t} + t + 2)\vec{j}.$$

The x-coordinate of the object is 3 when $3 = 3t^2$; thus $1 = t^2$. We are told that $0 \le t$, thus, the x-coordinate of the object is 3 when $t = +1$. When $t = +1$, the y-coordinate of the object is $4e^2 + 3$. We conclude that

when the x-coordinate of the object is 3, then the y-coordinate of the object is $4e^2 + 3$.