Math 241, Exam 2, Fall, 2022, Solutions

You should KEEP this piece of paper. Write everything on the **blank paper provided**. Return the problems **in order** (use as much paper as necessary), use **only one side** of each piece of paper. Number your pages and write your name on each page. Take a picture of your exam (for your records) just before you turn the exam in. I will e-mail your grade and my comments to you. I will keep your exam. **Fold your exam in half** before you turn it in.

The exam is worth 50 points. Each problem is worth 10 points. **Make your work coherent, complete, and correct.** Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find an equation for the plane through the points** $P_1 = (2, 4, 5)$**,** $P_2 = (1, -2, 4)$**, and** $P_3 = (3, 2, 1)$**. Check your answer. Make sure it is correct.**

We calculate

$$\overrightarrow{P_2P_1} = \vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}} + \vec{\boldsymbol{k}} \quad \text{and} \quad \overrightarrow{P_1P_3} = \vec{\boldsymbol{i}} - 2\vec{\boldsymbol{j}} - 4\vec{\boldsymbol{k}};$$

thus

$$\overrightarrow{P_2P_1} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ 1 & 6 & 1 \\ 1 & -2 & -4 \end{vmatrix} = \begin{vmatrix} 6 & 1 \\ -2 & -4 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} 1 & 1 \\ 1 & -4 \end{vmatrix} \vec{\boldsymbol{j}} - \begin{vmatrix} 1 & 6 \\ 1 & -2 \end{vmatrix} \vec{\boldsymbol{k}}$$

$$-22\vec{\boldsymbol{i}} + 5\vec{\boldsymbol{j}} - 8\vec{\boldsymbol{k}}.$$

The plane through $(2, 4, 5)$ perpendicular to $-22\vec{\boldsymbol{i}} + 5\vec{\boldsymbol{j}} - 8\vec{\boldsymbol{k}}$ is

$$-22(x - 2) + 5(y - 4) - 8(z - 5) = 0.$$

The above equation cleans up to become

$$\boxed{-22x + 5y - 8z = -64}.$$

Check:

Plug $(2, 4, 5)$ into the proposed answer:

$$-22(2) + 5(4) - 8(5) = -64\checkmark.$$

Plug $(1, -2, 4)$ into the proposed answer:

$$-22(1) + 5(-2) - 8(4) = -64\checkmark.$$

Plug $(3, 2, 1)$ into the proposed answer:

$$-22(3) + 5(2) - 8(1) = -64\checkmark.$$

(2) **Put** $2x^2 - 4x + 3y^2 - 12y + 4z^2 + 8z + 2 = 0$ **in the form**

$$A(x - x_0)^2 + B(y - y_0)^2 + C(z - z_0)^2 = D,$$

where x_0**,** y_0**,** z_0**,** A**,** B**,** C**, and** D **are numbers.**

We re-write the original equation as

$$2(x^2 - 2x + \boxed{1}) + 3(y^2 - 4y + \boxed{4}) + 4(z^2 + 2z + \boxed{1}) = -2 + 2(\boxed{1}) + 3\boxed{4} + 4\boxed{1}.$$

$$\boxed{2(x-1)^2 + 3(y-2)^2 + 4(z+1)^2 = 16.}$$

(3) **Name, describe, and graph the set of all points in three-space which satisfy** $x^2 + y^2 - z^2 = 1$**.**

When $x = 0$ the equation is $y^2 - z^2 = 1$ and the graph is a hyperbola which contains the point $y = 1$ and $z = 0$.

When $y = 0$ the equation is $x^2 - z^2 = 1$ and the graph is a hyperbola which contains the point $x = 1$ and $z = 0$.

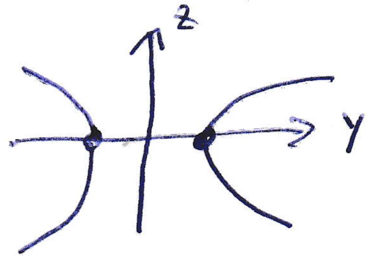
When $z = 0$ the equation is $x^2 + y^2 = 1$ and the graph is a circle.

The complete surface is a hyperboloid of one sheet. Draw the $y^2 - z^2 = 1$ in the yz-plane, then rotate this plane about the z-axis.

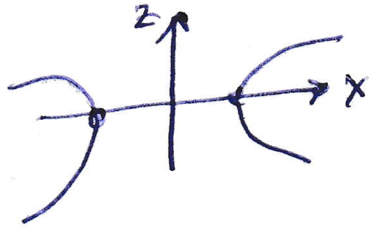
All four pictures appear on the next page.

Problem 3

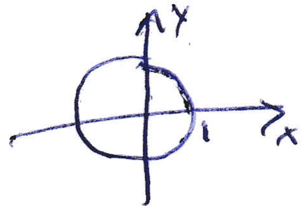
When X=0

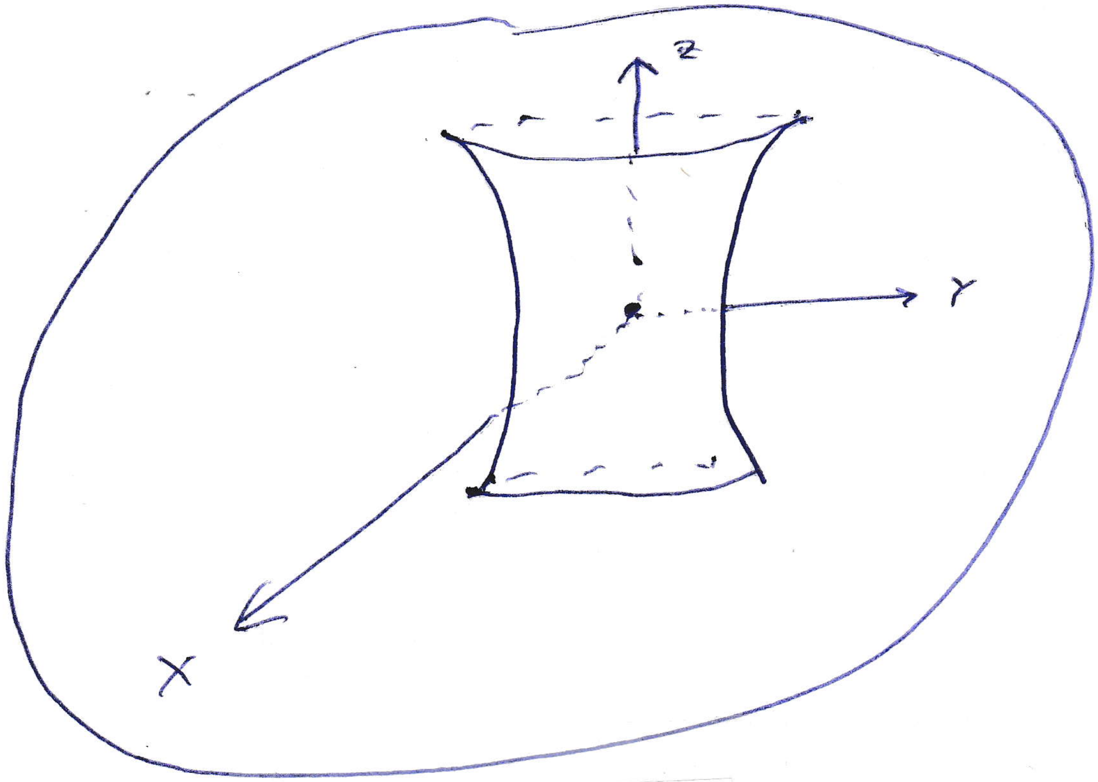


when y=0



when ~~z~~ =0





hyperboloid of one sheet.

(4) **An object travels on the xy-plane. The position vector, $\overrightarrow{\boldsymbol{r}}(t)$, of the object satisfies**

$$\begin{aligned}\overrightarrow{\boldsymbol{r}}''(t) &= 8e^{2t}\overrightarrow{\boldsymbol{i}} + e^t\overrightarrow{\boldsymbol{j}}\\ \overrightarrow{\boldsymbol{r}}'(t) &= 4e^{2t}\overrightarrow{\boldsymbol{i}} + e^t\overrightarrow{\boldsymbol{j}}\\ \overrightarrow{\boldsymbol{r}}'(0) &= 4\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}}\\ \overrightarrow{\boldsymbol{r}}(0) &= 2\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}}.\end{aligned}$$

What is the x-coordinate of the object when the y-coordinate is 3?

Integrate $\overrightarrow{\boldsymbol{r}}''(t)$ to learn $\overrightarrow{\boldsymbol{r}}'(t) = 4e^{2t}\overrightarrow{\boldsymbol{i}} + e^t\overrightarrow{\boldsymbol{j}} + \overrightarrow{c}_1$, for some constant vector $\overrightarrow{c}_1$. Plug in $t = 0$ to learn that

$$4\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} = \overrightarrow{\boldsymbol{r}}'(0) = 4e^0\overrightarrow{\boldsymbol{i}} + e^0\overrightarrow{\boldsymbol{j}} + \overrightarrow{c}_1 = 4\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} + \overrightarrow{c}_1.$$

Thus, $\overrightarrow{c}_1 = 0$ and $\overrightarrow{\boldsymbol{r}}'(t) = 4e^{2t}\overrightarrow{\boldsymbol{i}} + e^t\overrightarrow{\boldsymbol{j}}$. Integrate again to learn that

$$\overrightarrow{\boldsymbol{r}}(t) = 2e^{2t}\overrightarrow{\boldsymbol{i}} + e^t\overrightarrow{\boldsymbol{j}} + \overrightarrow{c}_2$$

for some constant vector $\overrightarrow{c}_2$. Plug in $t = 0$ to learn that

$$2\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} = \overrightarrow{\boldsymbol{r}}(0) = 2e^0\overrightarrow{\boldsymbol{i}} + e^0\overrightarrow{\boldsymbol{j}} + \overrightarrow{c}_2 = 2\overrightarrow{\boldsymbol{i}} + \overrightarrow{\boldsymbol{j}} + \overrightarrow{c}_2.$$

Thus, $\overrightarrow{c}_2 = 0$ and $\overrightarrow{\boldsymbol{r}}(t) = 2e^{2t}\overrightarrow{\boldsymbol{i}} + e^t\overrightarrow{\boldsymbol{j}}$.

The y-coordinate of the object is 3 when $e^t = 3$. The x-coordinate of the object is always $2e^{2t} = 2(e^t)^2$. When $e^t = 3$, the x-coordinate of the object is

$$\boxed{2(3^2)}.$$

(5) **Let $f(x,y) = \sqrt{x\cos(2xy) + 3x^2y^3}$. Find $\frac{\partial f}{\partial x}$.**

We calculate that $\frac{\partial f}{\partial x}$ is equal to

$$\boxed{\frac{-2xy\sin(2xy) + \cos(2xy) + 6xy^3}{2\sqrt{x\cos(2xy) + 3x^2y^3}}}.$$