Math 241, Exam 2, Fall 2019

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Wednesday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find the point on the plane** $x+2y+3z=25$ **which is closest to the point** $(\frac{11}{2},0,\frac{27}{2})$**.**

A vector perpendicular to the plane is $\vec{\boldsymbol{i}}+2\vec{\boldsymbol{j}}+3\vec{\boldsymbol{k}}$. The line perpendicular to the plane which passes through $(\frac{11}{2},0,\frac{27}{2})$ is

$$x-\tfrac{11}{2}=t,\quad y=2t,\quad z-\tfrac{27}{2}=3t.$$

The line and the plane meet when

$$(t+\tfrac{11}{2})+2(2t)+3(3t+\tfrac{27}{2})=25$$

$$\begin{aligned}14t&=25-\frac{11}{2}-\frac{81}{2}\\ 14t&=25-46\\ t&=\frac{-21}{14}\\ t&=\frac{-3}{2}.\end{aligned}$$

The point of intersection is

$$x=\frac{-3}{2}+\frac{11}{2},\quad y=-3,\quad z=\frac{-9}{2}+\frac{27}{2}.$$

The point on the plane closest to $(\frac{11}{2},0,\frac{27}{2})$ is $\boxed{(4,-3,9)}$.

(2) **Write** $4x^2+9y^2+z^2-8x+36y-6z+13=0$ **in the form**

$$\frac{(x-x_0)^2}{a^2}+\frac{(y-y_0)^2}{b^2}+\frac{(z-z_0)^2}{c^2}=1,$$

where x_0, y_0, z_0, a, b, and c are numbers.

Complete the square:

$$4x^2 + 9y^2 + z^2 - 8x + 36y - 6z + 13 = 0$$

$$4x^2 - 8x + 9y^2 + 36y + z^2 - 6z + 13 = 0$$

$$4(x^2 - 2x + 1) + 9(y^2 + 4y + 4) + z^2 - 6z + 9 = -13 + 4 + 36 + 9$$

$$4(x-1)^2 + 9(y+2)^2 + (z-3) = 36$$

$$\boxed{\frac{(x-1)^2}{9} + \frac{(y+2)^2}{4} + \frac{(z-3)^2}{36} = 1.}$$

(3) **Describe, graph, and name the graph of $y^2 - x^2 - z^2 = 1$ in 3-space.**

See the picture page.

(4) **An object starts at the origin with velocity $4\vec{\boldsymbol{i}} + 8\vec{\boldsymbol{j}}$. The acceleration of the object at time t is $\vec{\boldsymbol{r}}''(t) = 2e^t\vec{\boldsymbol{i}} + 16e^{2t}\vec{\boldsymbol{j}}$. What is the x-coordinate of the object when the y=coordinate is 12?**

Integrate to learn that $\vec{\boldsymbol{r}}'(t) = 2e^t\vec{\boldsymbol{i}} + 8e^{2t}\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1$. Plug in $t = 0$ to learn

$$4\vec{\boldsymbol{i}} + 8\vec{\boldsymbol{j}} = \vec{\boldsymbol{r}}'(0) = 2\vec{\boldsymbol{i}} + 8\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_1.$$

So, $\vec{\boldsymbol{c}}_1 = 2\vec{\boldsymbol{i}}$,

$$\vec{\boldsymbol{r}}'(t) = (2e^t + 2)\vec{\boldsymbol{i}} + 8e^{2t}\vec{\boldsymbol{j}}.$$

Integrate again to learn

$$\vec{\boldsymbol{r}}(t) = (2e^t + 2t)\vec{\boldsymbol{i}} + 4e^{2t}\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_2.$$

Plug in $t = 0$ to learn

$$0 = \vec{\boldsymbol{r}}(0) = 2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} + \vec{\boldsymbol{c}}_2.$$

So, $\vec{\boldsymbol{c}}_2 = -2\vec{\boldsymbol{i}} - 4\vec{\boldsymbol{j}}$ and

$$\vec{\boldsymbol{r}}(t) = (2e^t + 2t - 2)\vec{\boldsymbol{i}} + (4e^{2t} - 4)\vec{\boldsymbol{j}}.$$

The y-coordinate of the object is 12 when $4e^{2t} - 4 = 12$; thus, $4e^{2t} = 16$;

$$e^{2t} = 4$$

$$2t = \ln 4$$

$$t = (\ln 4)/2 = (2\ln 2)/2 = \ln 2.$$

The x-coordinate of the object is $2e^{\ln 2} + 2\ln 2 - 2 = 4 + \ln 4 - 2 = \boxed{2 + \ln 4}$ when the y-coordinate of the object is 12.

(5) **Find the equations of the LINE normal to $z = x^2 + y^2$, when $x = 1$ and $y = 2$.**

Gradients are perpendicular to level sets. View the surface as the level set $0 = x^2 + y^2 - z$. Thus, $\left(\vec{\nabla}(x^2 + y^2 - z)\right)|_{(1,2,5)}$ is perpendicular to $z = x^2 + y^2$ at the point where $x = 1$, $y = 2$, and $z = 1^2 + 2^2 = 5$. We compute

$$(\vec{\nabla}(x^2 + y^2 - z))|_{(1,2,5)} = (2x\vec{\boldsymbol{i}} + 2y\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}})|_{(1,2,5)} = 2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}.$$

The line through $(1, 2, 5)$ parallel to $2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}$ is

$$\boxed{x - 1 = 2t, \quad y - 2 = 4t, \quad z - 5 = -t.}$$