Math 241, Exam 2, Fall, 2018

Write everything on the blank paper provided. **YOU SHOULD KEEP THIS PIECE OF PAPER.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Thursday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Describe, graph, and name** $9x^2 + 4y^2 + z^2 = 36$ **in 3-space.**

See the picture page.

(2) **Do the lines**

$$\begin{cases} x = 5 + t \\ y = 6 + t \\ z = 7 + t \end{cases} \quad \textbf{and} \quad \begin{cases} x = 7 - 2s \\ y = -7 + 3s \\ z = s \end{cases}$$

intersect? If so, where. If not, why not?

We look for times t_0 and s_0 with the property that

$$x(t_0) = x(s_0), \quad y(t_0) = y(s_0), \quad z(t_0) = z(s_0)$$

where the left x, y, z refer to the first line and the second x, y, z refer to the second line. This is a system of three equations in two unknowns. It may or may not have a solution. At any rate, we try to solve

$$\begin{cases} 5 + t_0 = 7 - 2s_0 \\ 6 + t_0 = -7 + 3s_0 \\ 7 + t_0 = s_0. \end{cases}$$

We try to solve

$$\begin{cases} 5 + t_0 = 7 - 2(7 + t_0) \\ 6 + t_0 = -7 + 3(7 + t_0) \\ 7 + t_0 = s_0. \end{cases}$$

$$\begin{cases} 3t_0 = -12 \\ -8 = 2t_0 \\ 7 + t_0 = s_0. \end{cases}$$

Thus, $t_0 = -4$ and $s_0 = 3$.

$$\boxed{\text{The intersection point is } (1,2,3).}$$

(3) **An object is fired from the origin in the xy-plane at an angle α from the positive x-axis with an initial speed of v_0. The acceleration of the object is $-g\vec{\boldsymbol{j}}$. How high is the object when its x-coordinate is R?**

Let $\vec{\boldsymbol{r}}(t) = x(t)\vec{\boldsymbol{i}} + y(t)\vec{\boldsymbol{j}}$ be the position vector of the object at time t. We are told that $\vec{\boldsymbol{r}}''(t) = -g\vec{\boldsymbol{j}}$, $\vec{\boldsymbol{r}}'(0) = v_0 \cos\alpha\,\vec{\boldsymbol{i}} + v_0 \sin\alpha\,\vec{\boldsymbol{j}}$, and $\vec{\boldsymbol{r}}(0) = 0\vec{\boldsymbol{i}} + 0\vec{\boldsymbol{j}}$. We integrate to learn $\vec{\boldsymbol{r}}'(t) = -gt\vec{\boldsymbol{j}} + \vec{c_1}$. Plug in $t = 0$ to learn

$$v_0 \cos\alpha\,\vec{\boldsymbol{i}} + v_0 \cos\alpha\,\vec{\boldsymbol{j}} = \vec{\boldsymbol{r}}'(0) = \vec{c_1}.$$

So,

$$\vec{\boldsymbol{r}}'(t) = v_0 \cos\alpha\,\vec{\boldsymbol{i}} + (v_0 \sin\alpha - gt)\vec{\boldsymbol{j}}.$$

Integrate again to learn

$$\vec{\boldsymbol{r}}(t) = (v_0 \cos\alpha)t\,\vec{\boldsymbol{i}} + \big((v_0 \sin\alpha)t - gt^2/2\big)\vec{\boldsymbol{j}} + \vec{c_2}.$$

Plug in $t = 0$ to learn

$$0 = \vec{\boldsymbol{r}}(0) = \vec{c_2}.$$

Thus,

$$\vec{\boldsymbol{r}}(t) = (v_0 \cos\alpha)t\,\vec{\boldsymbol{i}} + \big((v_0 \sin\alpha)t - gt^2/2\big)\vec{\boldsymbol{j}}.$$

The x-coordinate of the object is R when

$$(v_0 \cos\alpha)t = R,$$

so $t = R/(v_0 \cos\alpha)$. When the x-coordinate is R, the y coordinate is

$$y(R/(v_0\cos\alpha)) = (v_0 \sin\alpha)(R/(v_0\cos\alpha)) - g\left(\frac{(R/(v_0\cos\alpha))^2}{2}\right)$$

$$= \boxed{R\tan\alpha - \frac{gR^2}{2v_0^2\cos^2\alpha}}.$$

(4) **Find the point on the curve**

$$\vec{\boldsymbol{r}}(t) = (5\sin t)\vec{\boldsymbol{i}} + (5\cos t)\vec{\boldsymbol{j}} + 12t\vec{\boldsymbol{k}}$$

at a distance 26π **units along the curve from the point** $(0,5,0)$ **in the direction of increasing arc length.**

Observe that $\vec{\boldsymbol{r}}(0) = (0,5,0)$. We first look for t_0 so that the length of the curve from $t=0$ to $t=t_0$ is 26π. That is,

$$26\pi = \int_0^{t_0} |\vec{\boldsymbol{r}}'(t)|dt = \int_0^{t_0} |(5\cos t)\vec{\boldsymbol{i}} + (-5\sin t)\vec{\boldsymbol{j}} + 12\vec{\boldsymbol{k}}|dt$$

$$= \int_0^{t_0} \sqrt{25\cos^2 t + 25\sin^2 + 144}dt = \int_0^{t_0} \sqrt{25(\cos^2 t + \sin^2) + 144}dt$$

$$= \int_0^{t_0} \sqrt{25+144}dt = \int_0^{t_0} \sqrt{169}dt = \int_0^{t_0} 13dt = 13t_0.$$

Solve $26\pi = 13t_0$ to see that $2\pi = t_0$. The point on the curve a distance 26π away from $(0,5,0)$ along the curve has position vector

$$\vec{\boldsymbol{r}}(2\pi) = (5\sin 2\pi)\vec{\boldsymbol{i}} + (5\cos 2\pi)\vec{\boldsymbol{j}} + 12(2\pi)\vec{\boldsymbol{k}} = 0\vec{\boldsymbol{i}} + 5\vec{\boldsymbol{j}} + 24\pi\vec{\boldsymbol{k}}.$$

The corresponding point on the curve is

$$\boxed{(0,5,24\pi).}$$

(5) **Express** $\vec{\boldsymbol{v}} = 4\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}}$ **as the sum of a vector parallel to**

$$\vec{\boldsymbol{b}} = -2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}$$

plus a vector perpendicular to $\vec{\boldsymbol{b}}$**. Check your answer. Make sure it is correct.**

Observe that $\vec{\boldsymbol{v}} = \text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}} + (\vec{\boldsymbol{v}} - \text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}})$ with $\text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}}$ parallel to $\vec{\boldsymbol{b}}$ and $(\vec{\boldsymbol{v}} - \text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}})$ perpendicular to $\vec{\boldsymbol{b}}$.

We compute

$$\text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}} = \frac{\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{v}}}{\vec{\boldsymbol{b}}\cdot\vec{\boldsymbol{b}}}\vec{\boldsymbol{b}} = \frac{(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}})\cdot(4\vec{\boldsymbol{i}} + \vec{\boldsymbol{j}})}{(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}})\cdot(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}})}(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}})$$

$$= \frac{-5}{13}(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}).$$

It follows that

$$(\vec{\boldsymbol{v}} - \text{proj}_{\vec{\boldsymbol{b}}}\vec{\boldsymbol{v}}) = \frac{52}{13}\vec{\boldsymbol{i}} + \frac{13}{13}\vec{\boldsymbol{j}} - (\frac{10}{13}\vec{\boldsymbol{i}} - \frac{15}{13}\vec{\boldsymbol{j}}) = \frac{42}{13}\vec{\boldsymbol{i}} + \frac{28}{13}\vec{\boldsymbol{j}}.$$

Thus $\vec{\boldsymbol{v}} = \frac{-5}{13}(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}) + (\frac{42}{13}\vec{\boldsymbol{i}} + \frac{28}{13}\vec{\boldsymbol{j}})$ with $\frac{-5}{13}(-2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}})$ parallel to $\vec{\boldsymbol{b}}$ and $(\frac{42}{13}\vec{\boldsymbol{i}} + \frac{28}{13}\vec{\boldsymbol{j}})$ perpendicular to $\vec{\boldsymbol{b}}$.