## Math 241, Exam 2, Fall, 2017 11:40 class

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Thrusday.

**No Calculators, Cell phones, computers, notes, etc.**

(1) **Express $\vec{v} = 4\vec{i} + \vec{j}$ as the sum of a vector parallel to $\vec{b} = -2\vec{i} + 3\vec{j}$ plus a vector perpendicular to $\vec{b}$. Check your answer. Make sure it is correct.**

Observe that $\vec{v} = \text{proj}_{\vec{b}}\,\vec{v} + (\vec{v} - \text{proj}_{\vec{b}}\,\vec{v})$ with $\text{proj}_{\vec{b}}\,\vec{v}$ parallel to $\vec{b}$ and $(\vec{v} - \text{proj}_{\vec{b}}\,\vec{v})$ perpendicular to $\vec{b}$.

We compute

$$\text{proj}_{\vec{b}}\,\vec{v} = \frac{\vec{b}\cdot\vec{v}}{\vec{b}\cdot\vec{b}}\vec{b} = \frac{(-2\vec{i}+3\vec{j})\cdot(4\vec{i}+\vec{j})}{(-2\vec{i}+3\vec{j})\cdot(-2\vec{i}+3\vec{j})}(-2\vec{i}+3\vec{j})$$

$$= \frac{-5}{13}(-2\vec{i}+3\vec{j}).$$

It follows that

$$(\vec{v} - \text{proj}_{\vec{b}}\,\vec{v}) = \frac{52}{13}\vec{i} + \frac{13}{13}\vec{j} - (\frac{10}{13}\vec{i} - \frac{15}{13}\vec{j}) = \frac{42}{13}\vec{i} + \frac{28}{13}\vec{j}.$$

Thus $\vec{v} = \frac{-5}{13}(-2\vec{i}+3\vec{j}) + (\frac{42}{13}\vec{i} + \frac{28}{13}\vec{j})$ with $\frac{-5}{13}(-2\vec{i}+3\vec{j})$ parallel to $\vec{b}$ and $(\frac{42}{13}\vec{i} + \frac{28}{13}\vec{j})$ perpendicular to $\vec{b}$.

(2) **Find the point on the line**

$$x = 2 + 3t, \quad y = 3 - t, \quad z = 1 + 2t$$

**which is nearest to the orign.**

Let $O = (0,0,0)$, $P = (2,3,1)$, and $Q = (a,b,c)$ be the point on the line which is closest to $O$. Observe that $P$ is on the line and $\vec{v} = 3\vec{i} - \vec{j} + 2\vec{k}$ is parallel to the line. Look at the picture to see that $\overrightarrow{PQ} = \text{proj}_{\vec{v}}\,\overrightarrow{PO}$. It follows that

$$(a-2)\vec{i} + (b-3)\vec{j} + (c-1)\vec{k} = \overrightarrow{PQ} = \frac{\vec{v}\cdot\overrightarrow{PO}}{\vec{v}\cdot\vec{v}}\vec{v}$$

$$= \frac{(3\vec{i} - \vec{j} + 2\vec{k}) \cdot (-2\vec{i} - 3\vec{j} - \vec{k})}{(3\vec{i} - \vec{j} + 2\vec{k}) \cdot (3\vec{i} - \vec{j} + 2\vec{k})}(3\vec{i} - \vec{j} + 2\vec{k})$$

$$= \frac{(-6+3-2)}{9+1+4}(3\vec{i} - \vec{j} + 2\vec{k}) = \frac{-5}{14}(3\vec{i} - \vec{j} + 2\vec{k})$$

The point $Q$ is $Q = (2 + \frac{-15}{14}, 3 + \frac{5}{14}, 1 - \frac{10}{14}) = \boxed{(\frac{13}{14}, \frac{47}{14}, \frac{4}{14})}$.

**Check.** First of all, $Q$ is the point on the line when $t = -5/14$. Furthermore, the vector $\overrightarrow{OQ}$ is $\frac{13}{14}\vec{i} + \frac{47}{14}\vec{j} + \frac{4}{14}\vec{k}$ and this vector is perpendicular to $\vec{v}$ because $(\frac{13}{14}\vec{i} + \frac{47}{14} + \vec{j}\,\frac{4}{14}\vec{k}) \cdot (3\vec{i} - \vec{j} + 2\vec{k}) = \frac{39-47+8}{14} = 0$

(3) **Graph and describe the set of points in 3-space which satisfy both of the equations**

$$z = 4 \quad \textbf{and} \quad (x-1)^2 + (y-2)^2 + (z-3)^2 = 16.$$

See the picture.

(4) **Let** $f(x,y) = 3x^2\sin(3y) + 7y\cos(2x)$**. Find** $\frac{\partial f}{\partial x}$**.**

$$\boxed{\frac{\partial f}{\partial x} = 6x\sin(3y) - 14y\sin(2x).}$$

(5) **An object is fired from the origin in the** $xy$**-plane at an angle** $\alpha$ **from the positive** $x$**-axis with an initial speed of** $v_0$**. The acceleration of the object is** $-g\vec{j}$**. How high is the object when its** $x$**-coordinate is** $R$**?**

Let $\vec{r}(t) = x(t)\vec{i} + y(t)\vec{j}$ be the position vector of the object at time $t$. We are told that $\vec{r}''(t) = -g\vec{j}$, $\vec{r}'(0) = v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}$, and $\vec{r}(0) = 0\vec{i} + 0\vec{j}$. We integrate to learn $\vec{r}'(t) = -gt\vec{j} + \vec{c_1}$. Plug in $t = 0$ to learn

$$v_0\cos\alpha\,\vec{i} + v_0\cos\alpha\,\vec{j} = \vec{r}'(0) = \vec{c_1}.$$

So,

$$\vec{r}'(t) = v_0\cos\alpha\,\vec{i} + (v_0\sin\alpha - gt)\vec{j}.$$

Integrate again to learn

$$\vec{r}(t) = (v_0\cos\alpha)t\,\vec{i} + \big((v_0\sin\alpha)t - gt^2/2\big)\vec{j} + \vec{c_2}.$$

Plug in $t = 0$ to learn

$$0 = \vec{r}(0) = \vec{c_2}.$$

Thus,

$$\vec{r}(t) = (v_0\cos\alpha)t\,\vec{i} + \big((v_0\sin\alpha)t - gt^2/2\big)\vec{j}.$$

The $x$-coordinate of the object is $R$ when

$$(v_0\cos\alpha)t = R,$$

so $t = R/(v_0 \cos\alpha)$. When the $x$-coordinate is $R$, the $y$ coordinate is

$$y(R/(v_0 \cos\alpha)) = (v_0 \sin\alpha)(R/(v_0 \cos\alpha)) - g\left(\frac{(R/(v_0 \cos\alpha))^2}{2}\right)$$

$$= \boxed{R\tan\alpha - \frac{gR^2}{2v_0^2 \cos^2\alpha}}.$$