Math 241, Exam 1, Spring, 2021

Write everything on the blank paper that you brought. There should be nothing on your desk except this exam, the blank paper that you brought, and a pen or pencil. When you are finished, send a picture of your solutions to

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The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

(1) **Find an equation for the plane through the points $P_1 = (-2, 3, 4)$, $P_2 = (3, -4, 5)$, and $P_3 = (2, -1, -8)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to our answer. We compute

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 5 & -7 & 1 \\ 4 & -4 & -12 \end{vmatrix} = (84+4)\,\vec{i} - (-60-4)\,\vec{j} + (-20+28)\,\vec{k}$$

$$= 88\,\vec{i} + 64\,\vec{j} + 8\,\vec{k}.$$

The plane through $(-2, 3, 4)$ perpendicular to $88\,\vec{i} + 64\,\vec{j} + 8\,\vec{k}$ is

$$88(x+2) + 64(y-3) + 8(z-4) = 0.$$

This plane is

$$11(x+2) + 8(y-3) + (z-4) = 0$$

or

$$\boxed{11x + 8y + z = 6.}$$

Check. Plug P_1 into the left side of the proposed answer to get

$$11(-2) + 8(3) + 4 = 6\checkmark.$$

Plug P_2 into the left side of the proposed answer to get

$$11(3) + 8(-4) + 5 = 6\checkmark.$$

Plug P_3 into the left side of the proposed answer to get

$$11(2) + 8(-1) - 8 = 6\checkmark.$$

(2) **Express $\overrightarrow{v} = 2\overrightarrow{i} - 3\overrightarrow{j}$ as the sum of a vector parallel to $\overrightarrow{b} = 3\overrightarrow{i} + 5\overrightarrow{j}$ and a vector perpendicular to $\overrightarrow{b}$. Check your answer. Make sure it is correct.**

We calculate

$$\text{proj}_{\overrightarrow{b}} \overrightarrow{v} = \frac{\overrightarrow{b} \cdot \overrightarrow{v}}{\overrightarrow{b} \cdot \overrightarrow{b}} \overrightarrow{b} = \frac{6 - 15}{9 + 25}(3\overrightarrow{i} + 5\overrightarrow{j}) = \frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j}).$$

We write

$$\overrightarrow{v} = \frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j}) + (\overrightarrow{v} - \frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j})).$$

We calculate

$$(\overrightarrow{v} - \frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j})) = \frac{95}{34}\overrightarrow{i} - \frac{57}{34}\overrightarrow{j}.$$

Thus,

$\overrightarrow{v} = \frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j}) + \left(\frac{95}{34}\overrightarrow{i} - \frac{57}{34}\overrightarrow{j}\right)$
with $\frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j})$ parallel to $\overrightarrow{b}$ and $\frac{95}{34}\overrightarrow{i} - \frac{57}{34}\overrightarrow{j}$ perpendicular to $\overrightarrow{b}$.

Check. It is clear that

$$\overrightarrow{v} = \frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j}) + \left(\frac{95}{34}\overrightarrow{i} - \frac{57}{34}\overrightarrow{j}\right)$$

and $\frac{-9}{34}(3\overrightarrow{i} + 5\overrightarrow{j})$ is parallel to $\overrightarrow{b}$. Observe also that

$$\overrightarrow{b} \cdot \left(\frac{95}{34}\overrightarrow{i} - \frac{57}{34}\overrightarrow{j}\right) = \frac{285}{34} - \frac{285}{34} = 0,$$

as expected.

(3) **Find parametric equations for the line which passes through the points $P = (1, 2, 3)$ and $Q = (3, 5, 7)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{PQ}$ is $2\overrightarrow{i} + 3\overrightarrow{j} + 4\overrightarrow{j}$. The line through $P = (1, 2, 3)$ parallel to $\overrightarrow{PQ} = 2\overrightarrow{i} + 3\overrightarrow{j} + 4\overrightarrow{j}$ is

$$\begin{cases} x = 1 + 2t \\ y = 2 + 3t \\ z = 3 + 4t \end{cases}$$

Check. The proposed answer gives P at $t = 0$ and the proposed answer gives Q at $t = 1$.

(4) **Find the point on $x + 2y + 3z = 6$ which is closest to $(-1, 3, 2)$.**

The line which passes through the point $(-1, 3, 2)$ and is perpendicular to the plane $x + 2y + 3z = 6$ is

$$\begin{cases} x = -1 + t \\ y = 3 + 2t \\ z = 2 + 3t \end{cases} \tag{1}$$

The answer is the point where the line (1) intersects the plane. This intersection occurs when

$$(-1 + t) + 2(3 + 2t) + 3(2 + 3t) = 6.$$

The intersection occurs when $t = -\frac{5}{14}$. The point of intersection is

$$\boxed{\left(\frac{-19}{14}, \frac{32}{14}, \frac{13}{14}\right)}.$$

Check. The proposed answer is on the plane and the vector from the proposed answer to the point $(-1, 3, 2)$ is perpendicular to the plane.

(5) **Write $2x^2 + 3y^2 - 8x + 12y + 15 = 0$ in the form $a(x - x_0)^2 + b(y - y_0)^2 = c$ for some numbers a, b, c, x_0, and y_0.**

The original equation may be rewritten as

$$2(x^2 - 4x + \boxed{4}) + 3(y^2 + 4y + \boxed{\boxed{4}} = -15 + 2\boxed{4} + 3\boxed{\boxed{4}}$$

or

$$\boxed{2(x - 2)^2 + 3(y + 2)^2 = 5.}$$