

Math 241, Exam 1, Spring, 2020 Solution

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please CIRCLE your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Monday.

No Calculators, Cell phones, computers, notes, etc.

- (1) **Find a system of parametric equations for the line through the points $P_1 = (7, 4, -2)$ and $P_2 = (1, -4, 1)$. Check your answer. Make sure it is correct.**

The line through $P_1 = (7, 4, -2)$ which is parallel to the vector $\overrightarrow{P_1P_2} = -6\vec{i} - 8\vec{j} + 3\vec{k}$ is

$$\begin{cases} x = 7 - 6t \\ y = 4 - 8t \\ z = -2 + 3t. \end{cases}$$

CHECK: If the proposed answer describes the position of an object at time t , then at $t = 0$, the object is standing at $(7, 4, -2) = P_1$ and at $t = 1$, the object is standing at $(1, -4, 1) = P_2$.

- (2) **Find an equation for the plane through the points $P_1 = (2, 3, 4)$, $P_2 = (3, 4, 5)$, and $P_3 = (1, 6, 8)$. Check your answer. Make sure it is correct.**

We compute

$$\begin{aligned} \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix} \vec{k} \\ &= \vec{i} - 5\vec{j} + 4\vec{k}. \end{aligned}$$

The plane through $P_1 = (2, 3, 4)$ perpendicular $\vec{i} - 5\vec{j} + 4\vec{k}$ is

$$(x - 2) - 5(y - 3) + 4(z - 4) = 0.$$

$$x - 5y + 4z = 3.$$

CHECK: Plug $P_1 = (2, 3, 4)$ into the proposed answer:

$$2 - 15 + 16 = 3 \checkmark$$

Plug $P_2 = (3, 4, 5)$ into the proposed answer:

$$3 - 20 + 20 = 3 \checkmark$$

Plug $P_3 = (1, 6, 8)$ into the proposed answer:

$$1 - 30 + 32 = 3 \checkmark$$

- (3) Express $\vec{v} = 1\vec{i} - 2\vec{j}$ as the sum of a vector parallel to $\vec{b} = 3\vec{i} + 4\vec{j}$ and a vector orthogonal to \vec{b} . Check your answer. Make sure it is correct.

We compute

$$\text{proj}_{\vec{b}} \vec{v} = \frac{\vec{b} \cdot \vec{v}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{3 - 8}{25} (3\vec{i} + 4\vec{j}) = -\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}.$$

We also compute

$$(\vec{v} - \text{proj}_{\vec{b}} \vec{v}) = +(\frac{8}{5}\vec{i} - \frac{6}{5}\vec{j}).$$

Thus

$$\vec{v} = (-\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j}) + (\frac{8}{5}\vec{i} - \frac{6}{5}\vec{j}),$$

where $(-\frac{3}{5}\vec{i} - \frac{4}{5}\vec{j})$ is parallel to \vec{b}
and $(\frac{8}{5}\vec{i} - \frac{6}{5}\vec{j})$ is perpendicular to \vec{b} .

CHECK: All three assertions are true.

- (4) **Parameterize the intersection of the two planes**

$$2x + 5y + 2z = 3 \quad \text{and} \quad 1x + y + 2z = 2.$$

Check your answer. Make sure it is correct.

Observe that the points $P = (0, 1/4, 7/8)$ and $Q = (1, 0, 1/2)$ are on both planes. The line through $Q = (1, 0, 1/2)$ parallel to $\overrightarrow{PQ} = \vec{i} - 1/4\vec{j} - 3/8\vec{k}$ is

$$\begin{cases} x = 1 + t \\ y = -(1/4)t \\ z = 1/2 - (3/8)t \end{cases}$$

CHECK: The proposed answer is on the plane $2x + 5y + 2z = 3$ because

$$2(1 + t) + 5(-1/4)t + 2(1/2 - (3/8)t) = 2 + 1 + t(2 - 5/4 - 3/4) = 3✓.$$

The proposed answer is on the plane $1x + y + 2z = 2$ because

$$(1 + t) - (1/4)t + 2(1/2 - (3/8)t) = 1 + 1 + t(1 - 1/4 - 3/4) = 2✓.$$

- (5) **Describe, name, and draw the set of all points in 3-space that satisfy both of the equations $x^2 + y^2 + z^2 = 25$ and $z = 4$.**

The picture is on a separate page. The set of points that satisfy both equations is the circle with center $(0, 0, 4)$ and radius 3 which lies in the plane $z = 4$. The plane $z = 4$ is parallel to the xy -plane but is 4 units higher.