Math 241, Exam 1, Spring, 2019

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Thursday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find a system of parametric equations for the line through the points $P_1 = (6, 3, -1)$ and $P_2 = (1, -4, 1)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2}$ is $-5\vec{i} - 7\vec{j} + 2\vec{k}$. The line through $(6, 3, -1)$ parallel to $-5\vec{i} - 7\vec{j} + 2\vec{k}$ is

$$\boxed{\begin{cases} x = 6 - 5t \\ y = 3 - 7t \\ z = -1 + 2t \end{cases}}$$

Check: When $t = 0$, our line sits at point $(6, 3, -1)$, which is P_1. When $t = 1$, our line sits at $(1, -4, 1)$, which is P_2.

(2) **Find an equation for the plane through the points $P_1 = (3, 4, 5)$, $P_2 = (4, 5, 6)$, and $P_3 = (2, 6, 8)$. Check your answer. Make sure it is correct.**

We calculate

$$\vec{N} = \overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \left| \begin{bmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & 1 \\ -1 & 2 & 3 \end{bmatrix} \right| = \left| \begin{bmatrix} 1 & 1 \\ 2 & 3 \end{bmatrix} \right| \vec{i} - \left| \begin{bmatrix} 1 & 1 \\ -1 & 3 \end{bmatrix} \right| \vec{j} + \left| \begin{bmatrix} 1 & 1 \\ -1 & 2 \end{bmatrix} \right| \vec{k}$$

$$= \vec{i} - 4\vec{j} + 3\vec{k}.$$

The plane through $P_1 = (3, 4, 5)$ perpendicular to $\vec{N} = \vec{i} - 4\vec{j} + 3\vec{k}$ is

$$1(x - 3) - 4(y - 4) + 3(z - 5) = 0,$$

which is also

$$\boxed{x - 4y + 3z = 2.}$$

Check: Plug $P_1 = (3,4,5)$ in to the proposed answer to get

$$2 - 16 + 15 = 2\checkmark.$$

Plug $P_2 = (4,5,6)$ in to the proposed answer to get

$$4 - 20 + 18 = 2\checkmark.$$

Plug $P_3 = (2,6,8)$ in to the proposed answer to get

$$2 - 24 + 24 = 2\checkmark.$$

(3) **Express $\overrightarrow{\boldsymbol{v}} = 2\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}}$ as the sum of a vector parallel to $\overrightarrow{\boldsymbol{b}} = 3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}}$ and a vector orthogonal to $\overrightarrow{\boldsymbol{b}}$. Check your answer. Make sure it is correct.**

We calculate

$$\operatorname{proj}_{\overrightarrow{\boldsymbol{b}}} \overrightarrow{\boldsymbol{v}} = \frac{\overrightarrow{\boldsymbol{b}} \cdot \overrightarrow{\boldsymbol{v}}}{\overrightarrow{\boldsymbol{b}} \cdot \overrightarrow{\boldsymbol{b}}} \overrightarrow{\boldsymbol{b}} = \frac{18}{25}(3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}})$$

$$\overrightarrow{\boldsymbol{v}} - \operatorname{proj}_{\overrightarrow{\boldsymbol{b}}} \overrightarrow{\boldsymbol{v}} = (2\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}}) - \frac{18}{25}(3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}}) = \frac{1}{25}(-4\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}}).$$

We see that $\frac{18}{25}(3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}})$ is parallel to $\overrightarrow{\boldsymbol{b}}$;
$\frac{1}{25}(-4\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}})$ is orthogonal to $\overrightarrow{\boldsymbol{b}}$; and
$\frac{18}{25}(3\overrightarrow{\boldsymbol{i}} + 4\overrightarrow{\boldsymbol{j}}) + \frac{1}{25}(-4\overrightarrow{\boldsymbol{i}} + 3\overrightarrow{\boldsymbol{j}}) = \overrightarrow{\boldsymbol{v}}$

Check: All three assertions are true and obvious.

(4) **Parameterize the intersection of the two planes**

$$3x + 6y + 2z = 3 \quad \textbf{and} \quad 2x + y + 2z = 2.$$

Check your answer. Make sure it is correct.

The point on both planes when $x = 0$ is $P = (0, \frac{2}{10}, \frac{9}{10})$. The point on both planes when $y = 0$ is $Q = (1,0,0)$. The vector from P to Q is $\overrightarrow{\boldsymbol{i}} - \frac{2}{10}\overrightarrow{\boldsymbol{j}} - \frac{9}{10}\overrightarrow{\boldsymbol{k}}$. The line through $(1,0,0)$ parallel to $\overrightarrow{\boldsymbol{i}} - \frac{2}{10}\overrightarrow{\boldsymbol{j}} - \frac{9}{10}\overrightarrow{\boldsymbol{k}}$ is

$$\boxed{\begin{cases} x = 1 + t \\ y = -\frac{2}{10}t \\ z = -\frac{9}{10}t \end{cases}}$$

Check: We see that our line is on the first plane because

$$3(1+t)+6(-\tfrac{2}{10}t)+2(-\tfrac{9}{10}t)=3+(3-\tfrac{12}{10}-\tfrac{18}{10})t=3\checkmark.$$

We see that our line is on the second plane because

$$2(1+t)+(-\frac{2}{10}t)+2(-\frac{9}{10}t)=2+(2-\frac{2}{10}-\frac{18}{10})t=2\checkmark.$$

(5) **Do the lines**

$$\begin{cases} x=s \\ y=-7+5s \\ z=7-3s \end{cases} \quad \textbf{and} \quad \begin{cases} x=-3-2t \\ y=6+4t \\ z=2-t \end{cases}$$

intersect? If so, where? If not, why not? Check your answer. Make sure it is correct.

We look for a time s_0 and a time t_0 so that some object moving on the first line stands on a particular point at time s_0 and some object moving on the second line is standing on the exact same point at time t_0. In other words, we want to find s_0 and t_0 so that all three equations

$$\begin{aligned} s_0 &= -3-2t_0 \\ -7+5s_0 &= 6+4t_0 \\ 7-3s_0 &= 2-t_0 \end{aligned}$$

hold simulaneously. So, we solve

$$\begin{aligned} s_0 &= -3-2t_0 \\ -7+5(-3-2t_0) &= 6+4t_0 \\ 7-3(-3-2t_0) &= 2-t_0 \end{aligned}$$

simulaneously.

So, we solve

$$\begin{aligned} s_0 &= -3-2t_0 \\ -7-15-6 &= (10+4)t_0 \\ 7+9-2 &= (-6-1)t_0 \end{aligned}$$

simulaneously. Thus, $t_0=-2$ and $s_0=1$. The point of intersection is $\boxed{(1,-2,4)}$.

Check: Plug $s_0=1$ into the first line:

$$x=1,\quad y=-7+5(1)=-2,\quad z=7-3(1)=4.$$

and $t_0=-2$ into the second line:

$$x=-3-2(-2)=1,\quad y=6+4(-2)=-2,\quad z=2-(-2)=4$$