Math 241, Exam 1, Fall 2019

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Monday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find a system of parametric equations for the line through the points** $P_1 = (1, 2, 3)$ **and** $P_2 = (7, 11, -1)$**. Check your answer. Make sure it is correct.**

Observe that $\overrightarrow{P_1P_2} = 6\,\vec{\boldsymbol{i}} + 9\,\vec{\boldsymbol{j}} + -4\,\vec{\boldsymbol{k}}$. The line through $(1, 2, 3)$ parallel to $6\,\vec{\boldsymbol{i}} + 9\,\vec{\boldsymbol{j}} - 4\,\vec{\boldsymbol{k}}$ is

$$x - 1 = 6t, \quad y - 2 = 9t, \quad z - 3 = -4t.$$

In other words,

$$\boxed{x = 1 + 6t, \quad y = 2 + 9t, \quad z = 3 - 4t.}$$

<u>**Check.**</u> If an object is travelling along our proposed answer, at time $t = 0$, the object is standing at $(1, 2, 3)$, which is P_1. At $t = 1$, the object is standing at $(7, 11, -1)$, which is P_2.

(2) **Find an equation for the plane through the points** $P_1 = (1, 2, 3)$**,** $P_2 = (-1, 0, 2)$**, and** $P_3 = (3, 1, 5)$**. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the desired plane. We compute

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -2 & -2 & -1 \\ 2 & -1 & 2 \end{vmatrix} = \begin{vmatrix} -2 & -1 \\ -1 & 2 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} -2 & -1 \\ 2 & 2 \end{vmatrix} \vec{\boldsymbol{j}} + \begin{vmatrix} -2 & -2 \\ 2 & -1 \end{vmatrix} \vec{\boldsymbol{k}}$$

$$= (-4 - 1)\,\vec{\boldsymbol{i}} - (-4 + 2)\,\vec{\boldsymbol{j}} + (2 + 4)\,\vec{\boldsymbol{k}} = -5\,\vec{\boldsymbol{i}} + 2\,\vec{\boldsymbol{j}} + 6\,\vec{\boldsymbol{k}}.$$

The plane through the point $(1,2,3)$ perpendicular to the vector $-5\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 6\vec{\boldsymbol{k}}$ is

$$-5(x-1)+2(y-2)+6(z-3)=0,$$

which is the same as

$$\boxed{-5x+2y+6z=17.}$$

Check. Plug $P_1=(1,2,3)$ into the proposed answer:

$$-5(1)+2(2)+6(3)=17\ \checkmark.$$

Plug $P_2=(-1,0,2)$ into the proposed answer:

$$-5(-1)+2(0)+6(2)=17\ \checkmark.$$

Plug $P_3=(3,1,5)$ into the proposed answer:

$$-5(3)+2(1)+6(5)=17\ \checkmark.$$

(3) **Express $\vec{\boldsymbol{v}} = \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}}$ as the sum of a vector parallel to $\vec{\boldsymbol{b}} = 3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}$ and a vector orthogonal to $\vec{\boldsymbol{b}}$. Check your answer. Make sure it is correct.**

Observe that

$$\begin{aligned}
\vec{\boldsymbol{v}} &= \operatorname{proj}_{\vec{\boldsymbol{b}}} \vec{\boldsymbol{v}} + (\vec{\boldsymbol{v}} - \operatorname{proj}_{\vec{\boldsymbol{b}}} \vec{\boldsymbol{v}}) \\
&= \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{b}}}{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}}}\vec{\boldsymbol{b}} + (\vec{\boldsymbol{v}} - \frac{\vec{\boldsymbol{v}} \cdot \vec{\boldsymbol{b}}}{\vec{\boldsymbol{b}} \cdot \vec{\boldsymbol{b}}}\vec{\boldsymbol{b}}) \\
&= \frac{11}{25}(3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}) + \frac{1}{25}\Big((25\vec{\boldsymbol{i}} + 50\vec{\boldsymbol{j}}) - (33\vec{\boldsymbol{i}} + 44\vec{\boldsymbol{j}})\Big) \\
&= \frac{11}{25}(3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}) + \frac{1}{25}(-8\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}}).
\end{aligned}$$

Thus,

$\vec{\boldsymbol{v}}$ is the sum of $\frac{11}{25}(3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}})$ plus $\frac{1}{25}(-8\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}})$ with $\frac{11}{25}(3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}})$ parallel to $\vec{\boldsymbol{b}}$ and $\frac{1}{25}(-8\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}})$ perpendicular to $\vec{\boldsymbol{b}}$.

Check. It is clear that

$$\frac{11}{25}(3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}}) + \frac{1}{25}(-8\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}}) = \vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}},$$

which is $\vec{\boldsymbol{v}}$.

It is clear that $\frac{11}{25}(3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}})$ is parallel to $\vec{\boldsymbol{b}}$.

We compute

$$3\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} \cdot \left(\frac{1}{25}(-8\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}})\right) = 0$$

to see that $\frac{1}{25}(-8\vec{\boldsymbol{i}} + 6\vec{\boldsymbol{j}})$ is perpendicular to $\vec{\boldsymbol{b}}$.

(4) **Find the point on the plane $5x + 3y - 7z = 73$ which is closest to the point $(1, 2, 3)$.**

The line which passes through $(1, 2, 3)$ and is perpendicular to the plane is

$$x - 1 = 5t, \quad y - 2 = 3t, \quad z - 3 = -7t.$$

This line hits the plane when

$$5(5t + 1) + 3(3t + 2) - 7(-7t + 3) = 73$$

$$(25 + 9 + 49)t + 5 + 6 - 21 = 73$$

$$83t = 83.$$

The line and the plane intersect when $t = 1$. The point of intersection is $(6, 5, -4)$.

$$\boxed{\text{The point on } 5x + 3y - 7z = 73 \text{ closest to } (1, 2, 3) \text{ is } (6, 5, -4)}$$

Check. The point $(6, 5, -4)$ is on the plane because $5(6)+3(5)-7(-4) = 73$. Furthermore, the vector which connects $(1, 2, 3)$ to $(6, 5, -4)$ is $5\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}} - 7\vec{\boldsymbol{k}}$, which is perpendicular to the plane $5x + 3y - 7z = 73$.

(5) **Write $4x^2 + 9y^2 + 36z^2 - 8x - 36y + 216z + 328 = 0$ in the form**

$$\frac{(x - x_0)^2}{a^2} + \frac{(y - y_0)^2}{b^2} + \frac{(z - z_0)^2}{c^2} = 1,$$

where x_0, y_0, z_0, a, b, and c are numbers.

The given equation has the same solutions as

$$4(x^2-2x+\boxed{1})+9(y^2-4y+\boxed{4})+36(z^2+6z+\boxed{\boxed{9}}) = -328+4\boxed{1}+9\boxed{4}+36\boxed{\boxed{9}}$$

$$4(x - 1)^2 + 9(y - 2)^2 + 36(z + 3)^2 = 36$$

$$\boxed{\frac{(x - 1)^2}{9} + \frac{(y - 2)^2}{4} + \frac{(z + 3)^2}{1} = 1}$$

Check. The proposed answer is in the right form. Just multiply it back out to see that it is correct. The proposed answer is equivalent to

$$4(x-1)^2 + 9(y-2)^2 + 36(z+3)^2 = 36$$

$$4(x^2 - 2x + 1) + 9(y^2 - 4y + 4) + 36(z^2 + 6z + 9) = 36$$
$$4x^2 - 8x + 4 + 9y^2 - 36y + 36 + 36z^2 + 216z + 36(9) = 36$$
$$4x^2 - 8x + 9y^2 - 36y + 36z^2 + 216z + 4 + 36 + 9(36) - 36 = 0$$

The number $4 + 36 + 9(36) - 36$ is equal to 328. ✓