Math 241, Exam 1, Fall, 2018

Write everything on the blank paper provided. **You should KEEP this piece of paper.** If possible: return the problems in order (use as much paper as necessary), use only one side of each piece of paper, and leave 1 square inch in the upper left hand corner for the staple. If you forget some of these requests, don't worry about it – I will still grade your exam.

The exam is worth 50 points. Each problem is worth 10 points. Please make your work coherent, complete, and correct. Please $\boxed{CIRCLE}$ your answer. Please **CHECK** your answer whenever possible.

The solutions will be posted later today.

The exams will be returned on Tuesday.

No Calculators, Cell phones, computers, notes, etc.

(1) **Find a system of parametric equations for the line through the points $P_1 = (2,3,-1)$ and $P_2 = (1,-1,1)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2}$ is $-\vec{\boldsymbol{i}} - 4\vec{\boldsymbol{j}} + 2\vec{\boldsymbol{k}}$. The line through P_1 parallel to $\overrightarrow{P_1P_2}$ is

$$\boxed{x = 2 - t, \quad y = 3 - 4t, \quad z = -1 + 2t.}$$

Check. The point on my answer when $t = 0$ is $(2,3,-1)$, which is P_1. The point on my answer when $t = 1$ is $(1,-1,1)$, which is P_2.

(2) **Find an equation for the plane through the points $P_1 = (3,4,5)$, $P_2 = (-1,5,7)$, and $P_3 = (1,6,8)$. Check your answer. Make sure it is correct.**

The vector $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the plane. We calculate

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \vec{\boldsymbol{i}} & \vec{\boldsymbol{j}} & \vec{\boldsymbol{k}} \\ -4 & 1 & 2 \\ -2 & 2 & 3 \end{vmatrix} = \begin{vmatrix} 1 & 2 \\ 2 & 3 \end{vmatrix} \vec{\boldsymbol{i}} - \begin{vmatrix} -4 & 2 \\ -2 & 3 \end{vmatrix} \vec{\boldsymbol{j}} + \begin{vmatrix} -4 & 1 \\ -2 & 2 \end{vmatrix} \vec{\boldsymbol{k}} = -\vec{\boldsymbol{i}} + 8\vec{\boldsymbol{j}} - 6\vec{\boldsymbol{k}}$$

The plane through $(3,4,5)$ perpendicular to $-\vec{\boldsymbol{i}} + 8\vec{\boldsymbol{j}} - 6\vec{\boldsymbol{k}}$ is

$$-(x-3) + 8(y-4) - 6(z-5) = 0$$

or

$$\boxed{-x + 8y - 6z = -1}$$

Check. The point $(3,4,5)$ satisfies the equation because

$$-3 + 32 - 30 = -1. \checkmark$$

The point $(-1, 5, 7)$ satisfies the equation because

$$1 + 40 - 42 = -1. \checkmark$$

The point $(1, 6, 8)$ satisfies the equation because

$$-1 + 48 - 48 = -1. \checkmark$$

(3) **Express $\vec{v} = \vec{i} + 3\vec{j}$ as the sum of a vector parallel to $\vec{b} = 1\vec{i} + 4\vec{j}$ and a vector orthogonal to $\vec{b}$. Check your answer. Make sure it is correct.**

We write $\vec{v} = \operatorname{proj}_{\vec{b}} \vec{v} + (\vec{v} - \operatorname{proj}_{\vec{b}} \vec{v})$, where $\operatorname{proj}_{\vec{b}} \vec{v}$ is parallel to $\vec{b}$ and $\vec{v} - \operatorname{proj}_{\vec{b}} \vec{v}$ is orthogonal to $\vec{b}$. We compute

$$\operatorname{proj}_{\vec{b}} \vec{v} = \frac{\vec{b} \cdot \vec{v}}{\vec{b} \cdot \vec{b}} \vec{b} = \frac{1 + 12}{1 + 16}(\vec{i} + 4\vec{j}) = \frac{13}{17}\vec{i} + \frac{52}{17}\vec{j}$$

and

$$\vec{v} - \operatorname{proj}_{\vec{b}} \vec{v} = \frac{1}{17}(17\vec{i} + 51\vec{j} - (13\vec{i} + 52\vec{j})) = \frac{1}{17}(4\vec{i} - \vec{j}).$$

$\vec{v}$ is equal to $\frac{13}{17}\vec{i} + \frac{52}{17}\vec{j}$ plus $\frac{1}{17}(4\vec{i} - \vec{j})$ with $\frac{13}{17}\vec{i} + \frac{52}{17}\vec{j}$ parallel to $\vec{b}$ and $\frac{1}{17}(4\vec{i} - \vec{j})$ orthogonal to $\vec{b}$.

Check. Please observe that all three of the assertions hold!!!

(4) **The set of all points on 3-space which satisfy $3x^2 - 6x + 3y^2 - 12y + 3 = 0$ is circular cylinder. What is the radius of this cylinder? What are the equations of the line in the center of the cylinder?**

We complete the square. Divide both sides by 3 to obtain

$$x^2 - 2x + \square + y^2 - 4y + \Delta = -1 + \square + \Delta$$

Put $(-2/2)^2 = 1$ into the boxes and $(-4/2)^2 = 4$ into the Δ's:

$$x^2 - 2x + 1 + y^2 - 4y + 4 = -1 + 1 + 4$$

$$(x - 1)^2 + (y - 2)^2 = 4$$

The radius of the cyllinder is 2 and the line in the center of the cylinder is $\begin{cases} x = 1 \\ y = 2. \end{cases}$

(Of course the center line is parallel to the z-axis; in particular z can be anything.)

(5) **What kind of geometric object is the intersection of the set of all points in 3-space which satisfy $x+y+3z=6$ and the set of all points in 3-space which satisfy $2x+y+z=3$? Parameterize this object.**

The geometric object is a line. We find two points on the line. When $x=0$, we solve $y+3z=6$ and $y+z=3$. Equation 1 minus equation two gives $2z=3$; hence $z=\frac{3}{2}$. It follows that $y=3-\frac{3}{2}=\frac{3}{2}$. Observe that $(0,\frac{3}{2},\frac{3}{2})$ does satisfy both equations.

When $z=0$, we solve $x+y=6$ and $2x+y=3$. Equation 1 minus equation 2 is $-x=3$ or $x=-3$. We compute that $y=6-x=6+3=9$. Observe that $(-3,9,0)$ does satisfy both equations. The vector from $(0,\frac{3}{2},\frac{3}{2})$ to $(-3,9,0)$ is $\overrightarrow{v}=-3\overrightarrow{i}+\frac{15}{2}\overrightarrow{j}-\frac{3}{2}\overrightarrow{k}$. The line through $(-3,9,0)$ and parallel to $\overrightarrow{v}$ is

$$\boxed{\begin{cases} x=-3-3t \\ y=9+\frac{15}{2}t \\ z=0-\frac{3}{2}t \end{cases}}$$

Check. Observe that our proposed answer is always on the first plane:

$$(-3-3t)+9+\tfrac{15}{2}t-3(\tfrac{3}{2}t)=6,\ \checkmark$$

and our proposed line is always on the second plane

$$2(-3-3t)+9+\tfrac{15}{2}t-\tfrac{3}{2}t=3\ \checkmark.$$