MATH 241, FALL 2024

CONTENTS

1. **Introductory remarks about the course.**

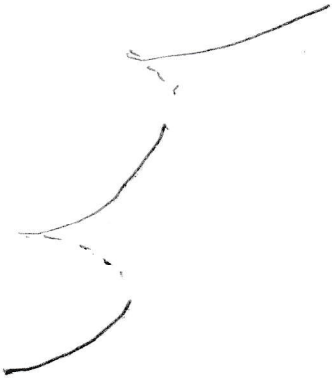
Here are some preliminary remarks about the course.

(1) My name is Professor Kustin.
(2) Be sure to look at the class website often. If you don't know the address, send an e-mail to me at kustin@math.sc.edu
(3) Quiz 1 on Wednesday, Aug. 28 is one of the assigned HW problems from 12.1 or 12.2.
(4) There is a list of assigned HW on my website.
(5) The HW is from
Thomas' Calculus: Early Transcendentals by George B. Thomas, Jr. Fifteenth Edition. The book was revised by Hass, Heil, Bogacki, and Weir. **Be sure to get access to these problems.** I can help, if necessary.
(6) There will be an Exam or Quiz essentially every two or three classes. The exams and quizzes will be given at the end of class. When you finish your quiz or exam, **take a picture of your solution for your records** and give me your answers. I will send my comments back by way of e-mail. I will return your paper when I next see you.
(7) If you miss an exam or quiz or do poorly on an exam or quiz; don't worry about it. There will be plenty more chances for you to demonstrate competence. Be sure to learn how to do missed or wrong problem correctly. (Talk to me about it; figure it out on your own; or use some other source.) I'll surely ask about it again.
(8) I have posted a typed version of the class lectures on my website. I encourage you to study them. In particular, if you miss a lecture, then I strongly encourage you to study the lecture notes. I will revise the typed notes as the class progresses.
(9) Ask about the things you don't understand.
    - Ask in class.
    - Ask during office hours.
    - Send me an e-mail.
    - Catch me before class.
    - Ask until you are satisfied.
(10) I want want you to learn the material. (If I am not happy with your work, I will complain vigorously.) I want you to earn a good grade. If you mess something up on a quiz or exam, I will surely ask you about it again. Get it right the second time (or the third time). The grading scheme is structured so that the early miss will not harm your final grade if you eventually figure it out. See "How your final grade will be calculated" at the end of the syllabus for full details.)

What we do. In Math 241 we do calculus in 3-space. We study curves, surfaces, and vector fields.

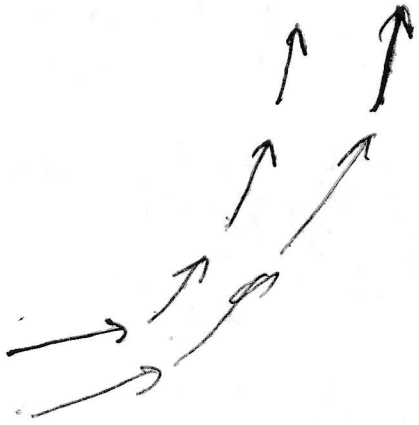
We study

curves



Surfaces

Vector fields!



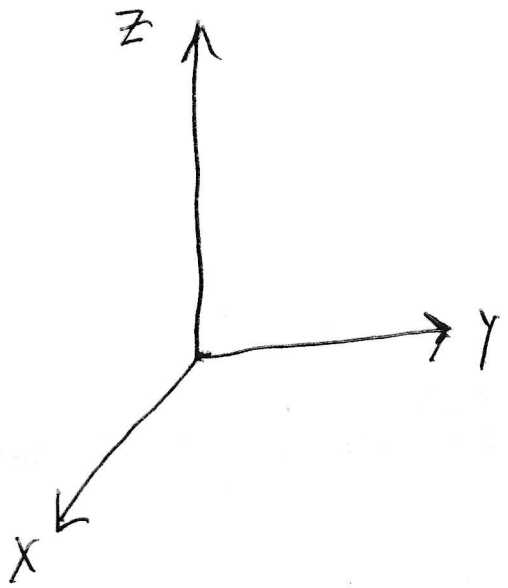
In a vector field, a directed line segment is put on each point in the plane (or in 3-space). A vector field might describe a fluid that is flowing or a force. The direction of the line segment is the direction a fluid is flowing at that point (or a force is acting at that point) and the length of the line segment is speed of the fluid flow (or the magnitude of the force). We do "max-min" problems for $z = f(x,y)$. We find areas, volumes, and masses for regions in the plane or 3-space.

2. Section 12.1. Introduction to 3-space.

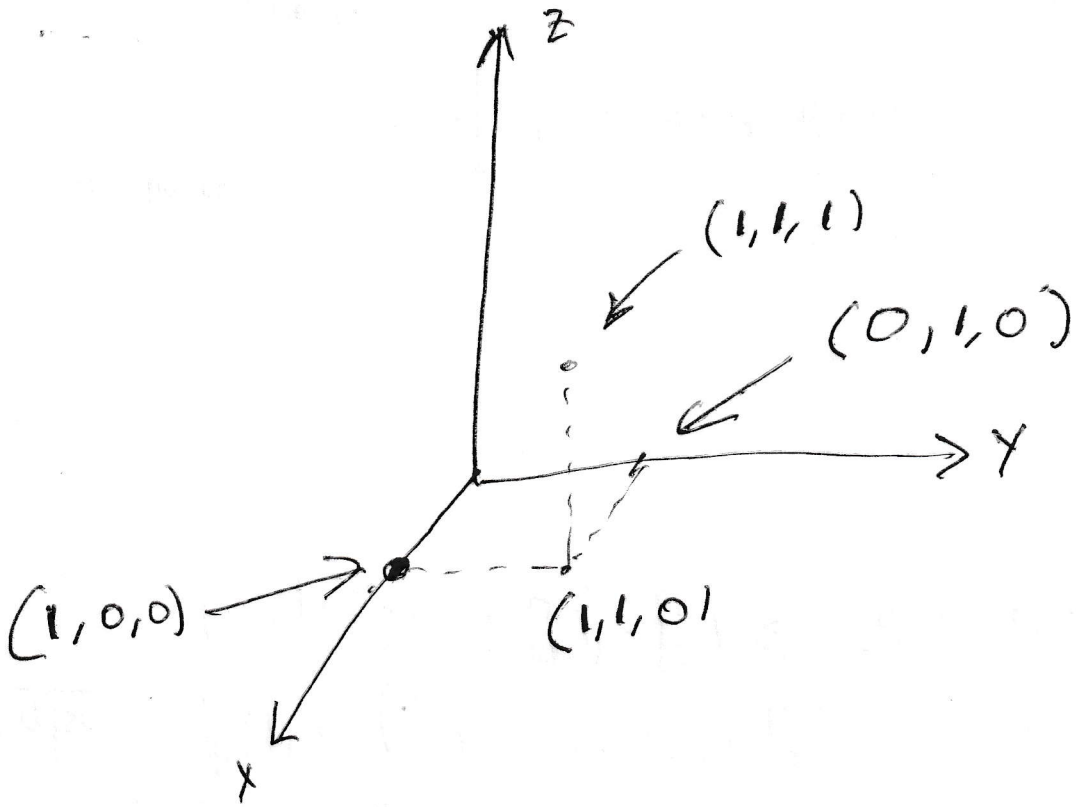
Most of our course takes place in 3-space. Mathematicians make the yz-plane be the plane of the blackboard and have the x-axis come into the classroom. One might wonder why Mathematicians do this. The reason is that in first semester calculus one studies (the graph of) $y = f(x)$ so the answer (which is y) is the height of the point $(x, f(x))$ (on the graph) above the domain (which is the x-axis). In third semester calculus one studies (the graph of) $z = f(x,y)$. Once again the answer (which is z) is the height of the point $(x,y,f(x,y))$ (on the graph) above the point in the domain (which is the xy-plane). If you can use the convention I am using, that is great. If you are already used to some other convention, I do not want to mess with your head, use the convention the convention that works for you; but do make sure to **label** the coordinate axes on any picture you draw for me!

On the next page, I drew the coordinate axes and plotted a few points in 3-space.

The coordinate axes



Some points in 3-space

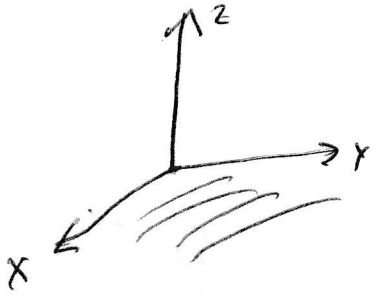


The equations of geometric objects in 3-space. What equation or equations is satisfied by each of the following geometric objects, but not satisfied by anything else in 3-space?

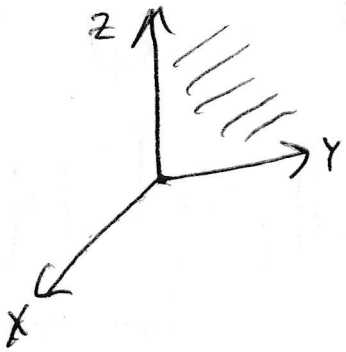
1. The xy-plane.
2. The yz-plane
3. The plane parallel to the yz-plane, but one unit in front of it.
4. The plane parallel to the xy-plane, but one unit above it.
5. The x-axis.
6. The y-axis.
7. The lines in the xy-plane that are parallel to the x-axis but are 2 units away from the x-axis.
8. The circle of radius 1 with center the origin in the yz-plane.
9. The sphere of radius 1 with center the origin.

The Equations of geometric objects page 1

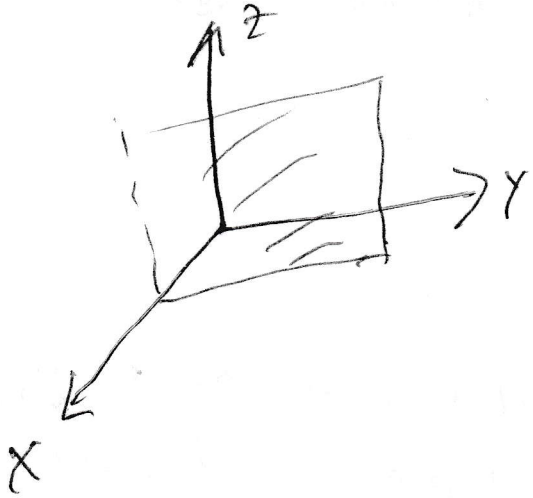
① The xy plane is $z=0$



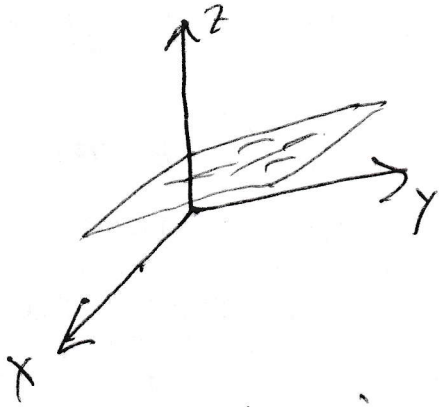
② The yz plane is $x=0$



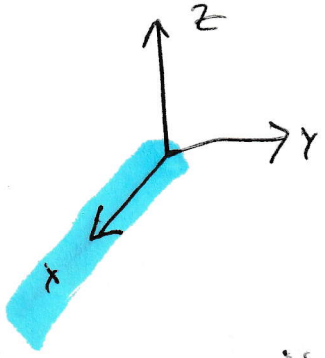
③ The plane parallel to the yz-plane and one unit in front of it is $x=1$.



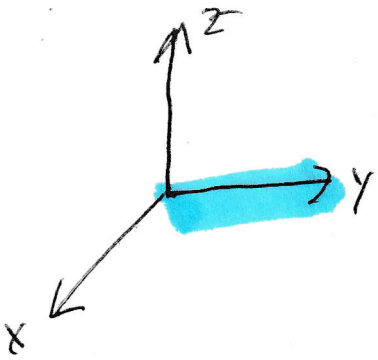
(4) The plane parallel to the xy-plane but one unit above it is $z=1$



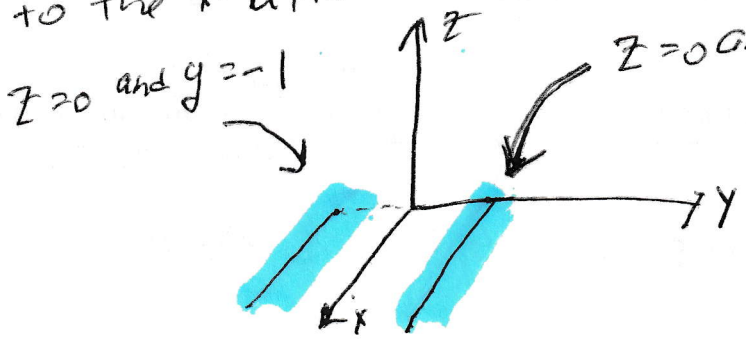
(5) The x-axis is $y=0$ and $z=0$



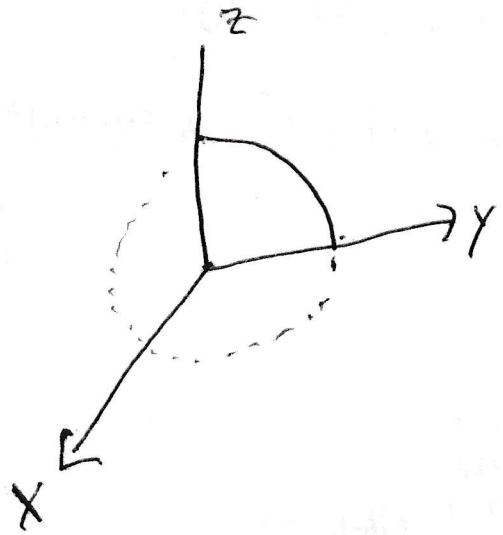
(6) The y-axis is $x=0$ and $z=0$



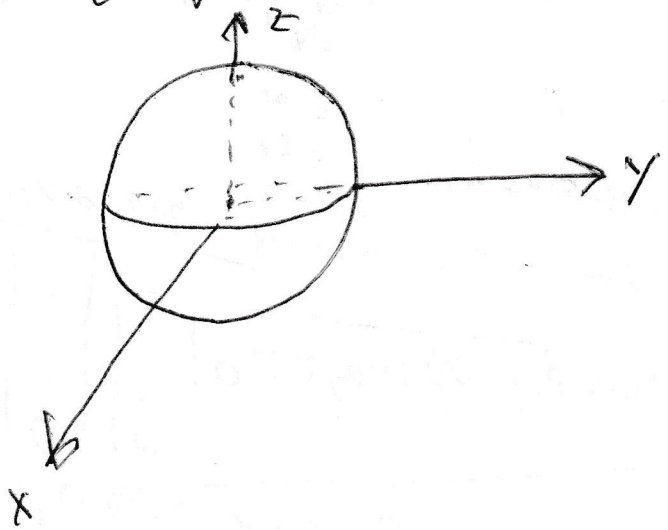
(7) The lines in the xyplane that are parallel to the x-axis but 2 units away



(8) The circle of radius 1 with center the origin in the yz-plane is $x=0$ and $y^2+z^2=1$



(9) The sphere of radius 1 with center the origin is $x^2+y^2+z^2=1$



Maybe we should think about the distance between two points in 3-space before we write down the equation of a sphere.

Fact 2.1. *If $P_1 = (x_1, y_1, z_1)$ and $P_2 = (x_2, y_2, z_2)$ are arbitrary points in 3-space, then the distance between P_1 and P_2 is*

$$\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2 + (z_2 - z_1)^2}.$$

Proof. Use the Pythagorean Theorem to see that the distance from (x_1, y_1, z_1) to (x_2, y_2, z_1) is $\sqrt{(x_2 - x_1)^2 + (y_2 - y_1)^2}$. (You know the Pythagorean Theorem. Even the Scarecrow in the Wizard of Oz knows it once he gets a brain.) Use the Pythagorean Theorem again to find the distance from (x_1, y_1, z_1) to (x_2, y_2, z_2). I put a nice picture on the next page. □

Fact 2.2. *If $P_0 = (x_0, y_0, z_0)$ is a point in 3-space r is a positive number, then the equation of the sphere with center P_0 and radius r is*

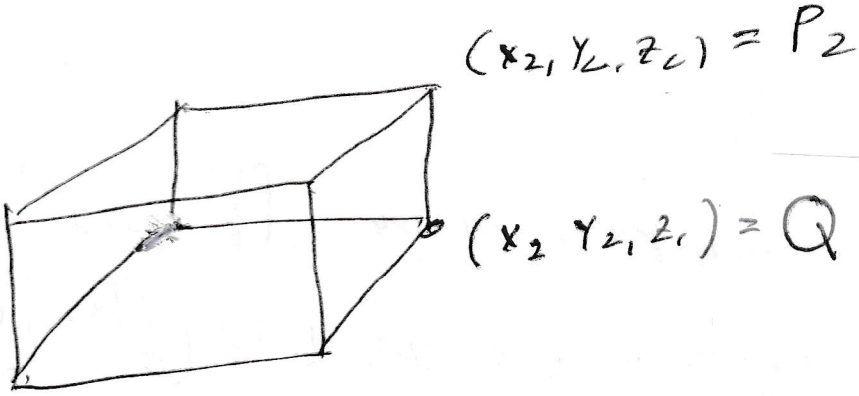
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

Proof. Let $P = (x, y, z)$ be an arbitrary point in 3-space. The point P is on our sphere if and only if the distance from P to P_0 equals r. Thus, the point P is on our sphere if and only

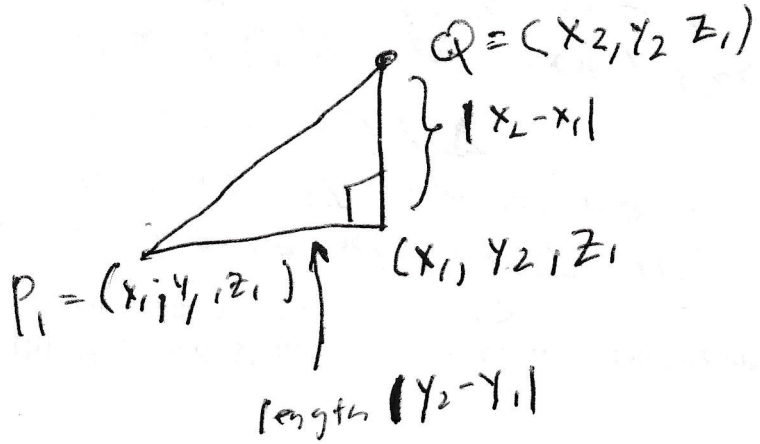
$$(x - x_0)^2 + (y - y_0)^2 + (z - z_0)^2 = r^2.$$

□

The distance from P_1 to P_2

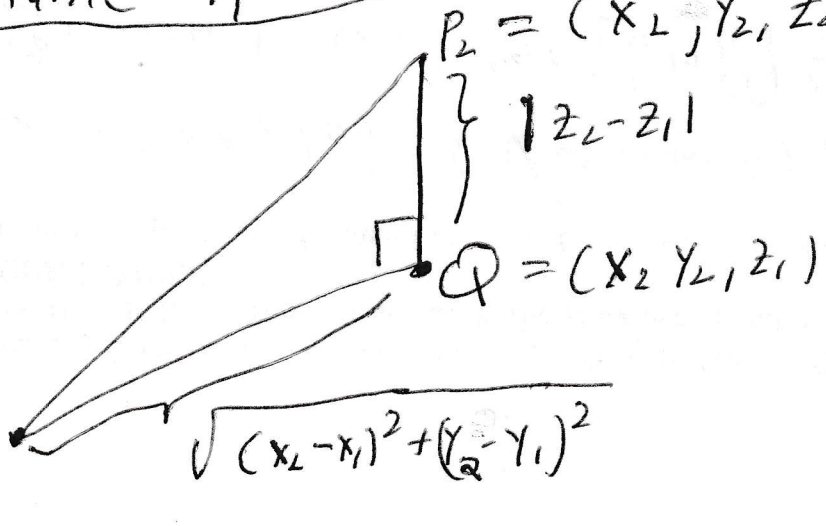


The distance from P_1 to Q by Pythagoras is



$$P_1Q = \sqrt{(x_2-x_1)^2 + (y_2-y_1)^2}$$

The distance P_1 to P_2 is $\sqrt{(x_2-x_1)^2 + (y_2-y_1)^2 + (z_2-z_1)^2}$



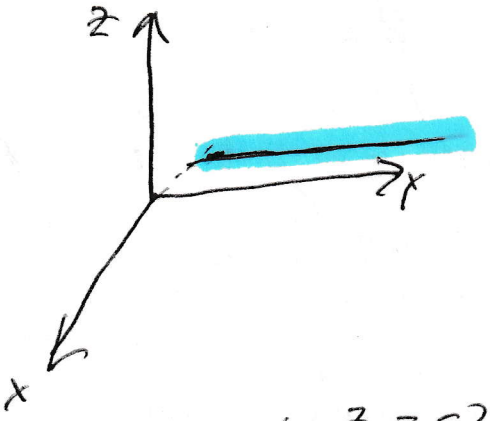
Describe, graph, and name the set of all points in 3-space which satisfy

10. $x = -1$ and $z = 0$,
11. $x^2 + y^2 = 4$ and $z = -2$,
12. $x^2 + y^2 + z^2 = 4$,
13. $y = x^2 + 4$.

Describe, graph, and name the set of all points in 3-space which satisfy

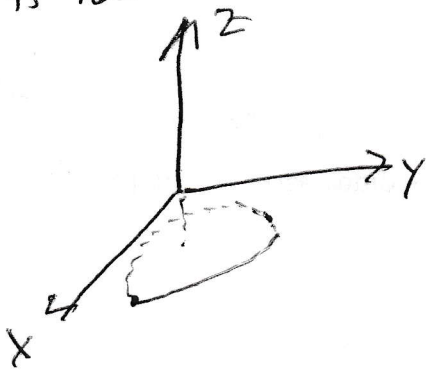
10. $X=-1$ and $Z=0$.

This is the line parallel to the y-axis, but one unit behind the y-axis



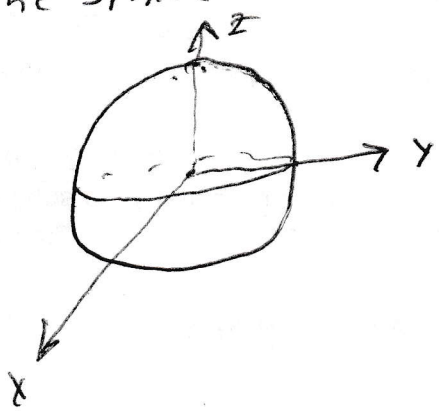
11. $X^2+Y^2=4$ and $Z=-2$

This is the circle of radius 2 with center $(0,0,-2)$ which sits in the plane parallel to the XY plane but 2 units lower



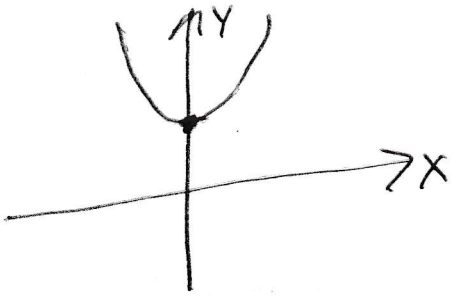
12. $X^2+Y^2+Z^2=4$

This is the sphere with radius 2 and center $(0,0,0)$

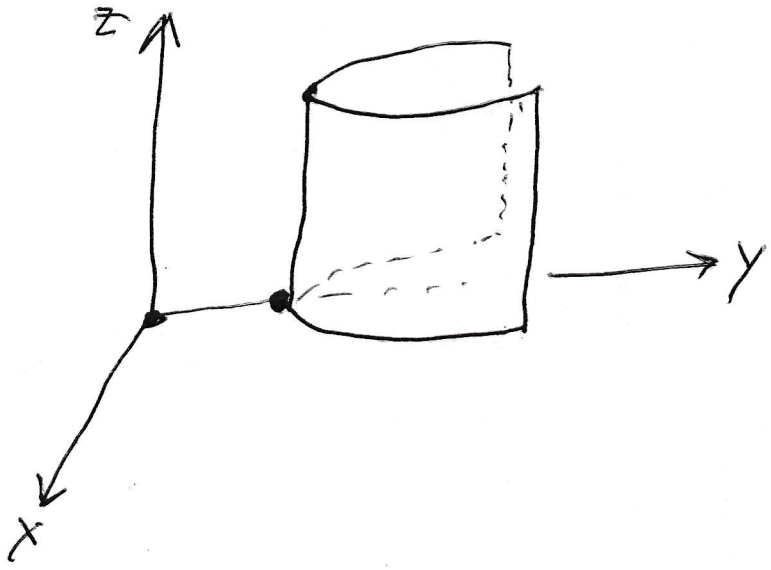


13 $y = x^2 + 4$

In the xy plane this is a parabola



In 3-space one gets the same picture for each z-- The resulting surface is called a "parabolic cylinder"



3. Section 12.2. Vectors.

A vector is a directed line segment. Two vectors are equal if they have the same length and the same direction.
The vector of length 1 in the direction of the positive x-axis is called $\vec{i}$;
the vector of length 1 in the direction of the positive y-axis is called $\vec{j}$;
the vector of length 1 in the direction of the positive z-axis is called $\vec{k}$.

Example 3.1. The vector from the point $(1,2)$ to the point $(3,5)$ is denoted

$$\overrightarrow{(1,2)(3,5)}.$$

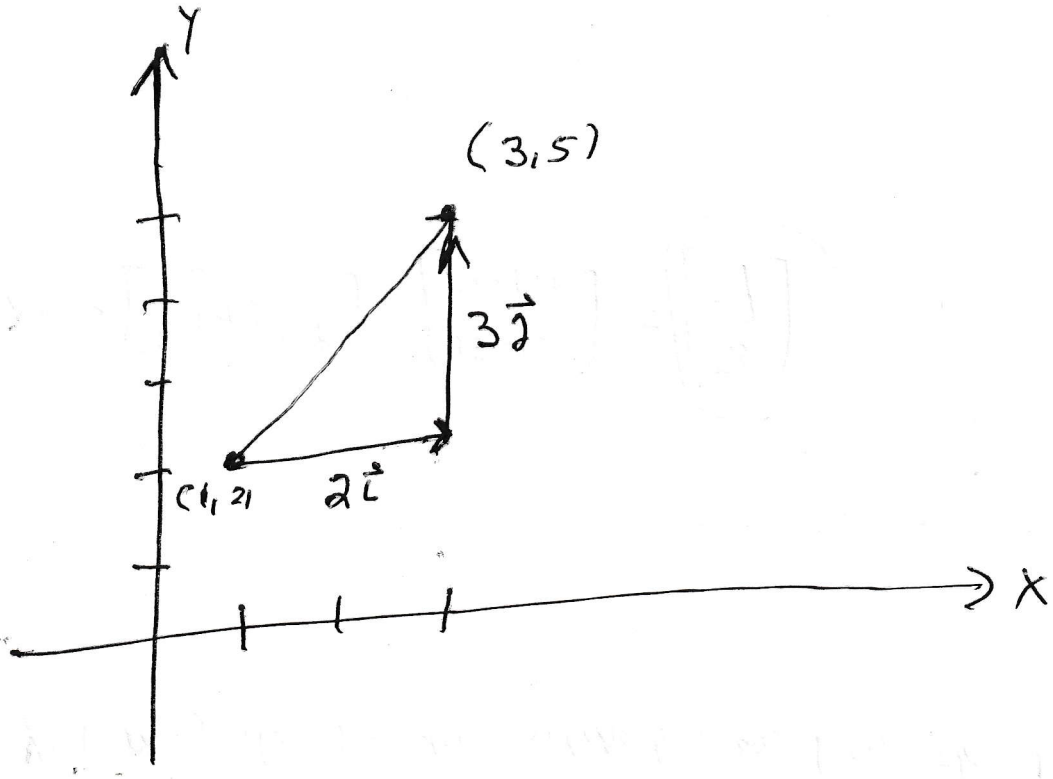
This vector is equal to $2\vec{i}+3\vec{j}$. There is a pretty picture of this vector on the next page. The vector $2\vec{i}+3\vec{j}$ may also be written as $\langle 2,3\rangle$. **Please use $\langle\ ,\ \rangle$ for vectors and $(\ ,\ )$ for points.** Vectors are different than points. You can move a vector without damaging it; but you can't move a point. The vector $2\vec{i}+3\vec{j}$ has length $\sqrt{2^2+3^2}=\sqrt{13}$. (The is just the Pythagorean Theorem. A vector is a line segment. One can use the Pythagorean Theorem to find the length of a line segment.) If $\vec{v}$ is the name of a vector, then $|\vec{v}|$ is the length of $\vec{v}$.

Example 3.2. Write the vector $\vec{v}=2\vec{i}+3\vec{j}$ in the form $c\vec{u}$, where $\vec{u}$ is a unit vector and c is a constant. (A unit vector is a vector of length one.) We already saw that $|\vec{v}|=\sqrt{13}$. It follows that $\frac{\vec{v}}{\sqrt{13}}$ is a unit vector and

$$\vec{v}=\sqrt{13}\left(\frac{2}{\sqrt{13}}\vec{i}+\frac{3}{\sqrt{13}}\vec{j}\right),$$

where $\frac{2}{\sqrt{13}}\vec{i}+\frac{3}{\sqrt{13}}\vec{j}$ is a unit vector and $\sqrt{13}$ is a constant.

The Vector from (1,2) to (3,5)



$$\overrightarrow{(1,2)(3,5)} = 2\vec{i} + 3\vec{j}$$

4. Section 12.3. Dot product.

Definition 4.1. If $\vec{v} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{w} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$ are vectors, then the dot product of $\vec{v}$ and $\vec{w}$ is the **number**

$$\vec{v} \cdot \vec{w} = a_1b_1 + a_2b_2 + a_3b_3.$$

Fact 4.2. *If $\vec{v}$ and $\vec{w}$ are vectors, then*

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos\theta,$$

where θ is the angle between $\vec{v}$ and $\vec{w}$. (*There is a picture of the angle between $\vec{v}$ and $\vec{w}$ on the next page.*)

Fact 4.2 is a quick consequence of the Law of Cosines. See, for example,

`https://en.wikipedia.org/wiki/Law_of_cosines`

Fact 4.2 explains why the dot product is important. We use it to find the angle between two vectors. In particular, it gives a **very easy** to decide when two vectors are perpendicular!

Example 4.3. Find the angle between $\overrightarrow{(2,3)(5,6)}$ and $\overrightarrow{(2,3)(1,7)}$. There is a picture on the next page. Let $\vec{v} = \overrightarrow{(2,3)(5,6)}$ and $\vec{w} = \overrightarrow{(2,3)(1,7)}$. Observe that

$$\vec{v} = 3\vec{i} + 3\vec{j} \quad \text{and} \quad \vec{w} = -\vec{i} + 4\vec{j}.$$

It follows that

$$\vec{v} \cdot \vec{w} = |\vec{v}||\vec{w}|\cos(\theta),$$

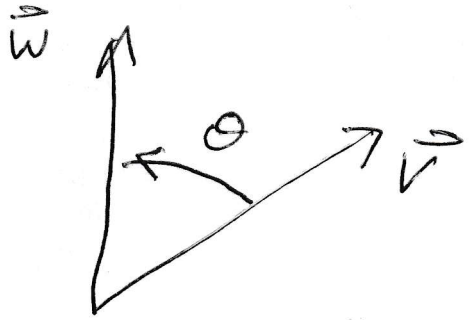
where θ is the angle between $\vec{v}$ and $\vec{w}$. Thus,

$$(3\vec{i} + 3\vec{j}) \cdot (-\vec{i} + 4\vec{j}) = |3\vec{i} + 3\vec{j}||-\vec{i} + 4\vec{j}|\cos\theta$$

$$-3 + 12 = 3\sqrt{2}\sqrt{17}\cos\theta$$

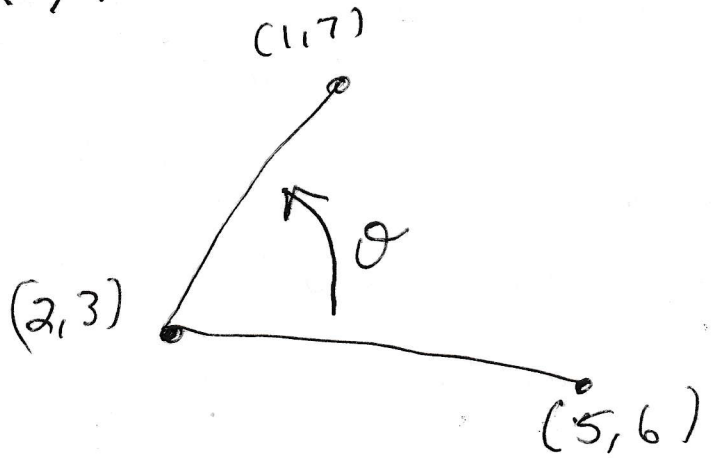
$$\boxed{\arccos\frac{9}{3\sqrt{2}\sqrt{17}} = \theta.}$$

The angle between two vectors



θ is the angle between $\vec{v}$ and $\vec{w}$

Example Find the angle between $\overrightarrow{(2,3)(5,6)}$ and $\overrightarrow{(2,3)(1,7)}$



The numbers are on the previous page

Question 4.4. *How does one use dot product to see that two vectors are perpendicular?*

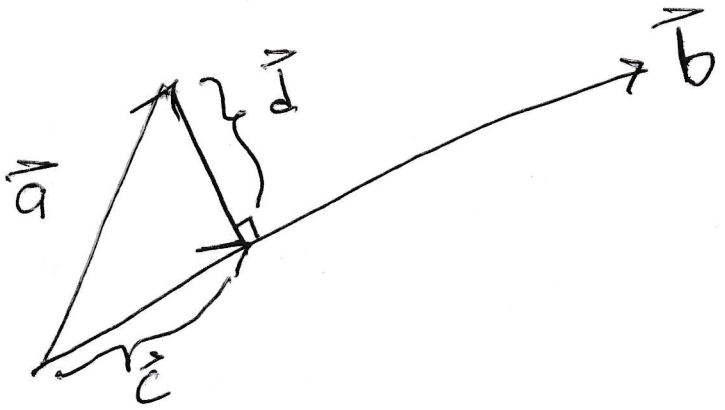
Answer. Two vectors are perpendicular if and only if their dot product is zero.

Examples 4.5. (a) The vectors $\overrightarrow{i}$ and $\overrightarrow{j}$ are perpendicular and $\overrightarrow{i} \cdot \overrightarrow{j} = 0$.

(b) The vectors $\overrightarrow{v} = \overrightarrow{i} + m\overrightarrow{j}$ and $\overrightarrow{w} = \overrightarrow{i} + \frac{-1}{m}\overrightarrow{j}$ are perpendicular for all non-zero m. Of course, every line that contains $\overrightarrow{v}$ is a line with slope m and every line that contains $\overrightarrow{w}$ is a line with slope $\frac{-1}{m}$. You have known for many years that if the slope of two lines multiply to -1, then the lines are perpendicular. The notion of dot product is a sophisticated version of this elementary fact about the slopes of perpendicular lines. There are two cool things about dot product. It works in 3-space as well as two-space! It also can be used to tell the angle between two lines even if the lines are not perpendicular.

Vector projection. Often one is given two vectors $\overrightarrow{a}$ and $\overrightarrow{b}$ and one wants to write $\overrightarrow{a}$ as the sum of two vectors $\overrightarrow{c}$ and $\overrightarrow{d}$, where $\overrightarrow{c}$ is parallel to $\overrightarrow{b}$ and $\overrightarrow{d}$ is perpendicular to $\overrightarrow{b}$. (There is a good picture on the next page.)

Vector Projection



$\vec{c}$ is called the projection of $\vec{a}$ on $\vec{b}$ and is denoted by $\text{proj}_{\vec{b}}\vec{a}$.

$$\boxed{\text{Proj}_{\vec{b}}\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}}$$

Notice that $\vec{d} = \vec{a} - \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}$

It is easy to see that

$$\vec{d}\cdot\vec{b} = \vec{a}\cdot\vec{b} - \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}\cdot\vec{b} = \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 0$$

hence $\vec{d}$ really is perpendicular to $\vec{b}$

For example if $\vec{a}$ is a force and $\vec{b}$ is the direction of motion, then $\vec{c}$ is the component of the force that is in the direction of motion and $\vec{d}$ is the component of the force that is perpendicular to the direction of motion.

Similarly, if $\vec{a}$ is an acceleration vector and $\vec{b}$ is the direction of motion, then $\vec{c}$ is the component of acceleration that is in the direction of motion and $\vec{d}$ is the component of acceleration that is perpendicular to the direction of motion.

The vector $\vec{c}$ is called the projection of $\vec{a}$ onto $\vec{b}$. This vector is usually denoted

$$\text{proj}_{\vec{b}}\,\vec{a}.$$

The formula for vector projection is

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}.$$

Please notice:

(a) The expression $\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}$ is a **number** and $\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}$ is a **vector**.

(b) The vector $\vec{d} = \vec{a} - \text{proj}_{\vec{b}}\,\vec{a}$.

(c) This is the correct answer because

(i) The vector $\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}$ is obviously parallel to $\vec{b}$.

(ii) The sum

$$\text{proj}_{\vec{b}}\,\vec{a} + \left(\vec{a} - \text{proj}_{\vec{b}}\,\vec{a}\right)$$

is obviously equal to $\vec{a}$.

(iii) The vector $\vec{a} - \text{proj}_{\vec{b}}\,\vec{a}$ is perpendicular to $\vec{b}$. This statement is not as obvious; but it is the point of the process. We check the statement by computing

$$\begin{aligned}\left(\vec{a} - \text{proj}_{\vec{b}}\,\vec{a}\right)\cdot\vec{b} &= \left(\vec{a} - \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}\right)\cdot\vec{b} = \vec{a}\cdot\vec{b} - \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}\cdot\vec{b}\\ &= \vec{a}\cdot\vec{b} - \vec{a}\cdot\vec{b} = 0.\end{aligned}$$

(d) The calculation (ciii) tells you how to figure out the formula for vector projection if you have forgotten it. We know that $\text{proj}_{\vec{b}}\,\vec{a}$ is equal to $c\,\vec{b}$ for some number c. We need only figure out the value of c. Well, we have to choose c so that "the left over vector" $\vec{a} - c\,\vec{b}$ is perpendicular to $\vec{b}$. We need only choose c so that

$$(\vec{a} - c\,\vec{b})\cdot\vec{b} = 0.$$

We want

$$\vec{a}\cdot\vec{b} - c\,\vec{b}\cdot\vec{b} = 0.$$

We see that c must be $\frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}$. We conclude that $\text{proj}_{\vec{b}}\,\vec{a}$ must be

$$\text{proj}_{\vec{b}}\,\vec{a} = \frac{\vec{a}\cdot\vec{b}}{\vec{b}\cdot\vec{b}}\,\vec{b}.$$

Example 4.6. Express $\overrightarrow{v} = 1\overrightarrow{i} - 2\overrightarrow{j}$ as the sum of a vector parallel to $\overrightarrow{b} = 3\overrightarrow{i} + 4\overrightarrow{j}$ and a vector orthogonal to $\overrightarrow{b}$.

We compute

$$\text{proj}_{\overrightarrow{b}}\overrightarrow{v} = \frac{\overrightarrow{b}\cdot\overrightarrow{v}}{\overrightarrow{b}\cdot\overrightarrow{b}}\overrightarrow{b} = \frac{3-8}{25}(3\overrightarrow{i} + 4\overrightarrow{j}) = -\frac{3}{5}\overrightarrow{i} - \frac{4}{5}\overrightarrow{j}.$$

We also compute

$$(\overrightarrow{v} - \text{proj}_{\overrightarrow{b}}\overrightarrow{v}) = +(\frac{8}{5}\overrightarrow{i} - \frac{6}{5}\overrightarrow{j}).$$

Thus

$$\boxed{\begin{array}{l}\overrightarrow{v} = (-\frac{3}{5}\overrightarrow{i} - \frac{4}{5}\overrightarrow{j}) + (\frac{8}{5}\overrightarrow{i} - \frac{6}{5}\overrightarrow{j}),\\ \text{where } (-\frac{3}{5}\overrightarrow{i} - \frac{4}{5}\overrightarrow{j}) \text{ is parallel to } \overrightarrow{b}\\ \text{and } (\frac{8}{5}\overrightarrow{i} - \frac{6}{5}\overrightarrow{j}) \text{ is perpendicular to } \overrightarrow{b}.\end{array}}$$

Notice that $(-\frac{3}{5}\overrightarrow{i} - \frac{4}{5}\overrightarrow{j})$ is parallel to $\overrightarrow{b}$ because $(-\frac{3}{5}\overrightarrow{i} - \frac{4}{5}\overrightarrow{j})$ is equal to $(\frac{-1}{5})\overrightarrow{b}$;

$$(-\frac{3}{5}\overrightarrow{i} - \frac{4}{5}\overrightarrow{j}) + (\frac{8}{5}\overrightarrow{i} - \frac{6}{5}\overrightarrow{j}) = \overrightarrow{i} - 2\overrightarrow{j} = \overrightarrow{v};$$

and

$$(\frac{8}{5}\overrightarrow{i} - \frac{6}{5}\overrightarrow{j})\cdot(3\overrightarrow{i} + 4\overrightarrow{j}) = \frac{24}{5} - \frac{24}{5} = 0.$$

5. Section 12.4. Cross product.

If $\vec{a}$ and $\vec{b}$ are vectors in $\mathbb{R}^3$, then $\vec{a} \times \vec{b}$ is a vector in $\mathbb{R}^3$. (One calls "$\vec{a} \times \vec{b}$" the cross product of $\vec{a}$ and $\vec{b}$.)

Here are three methods for computing cross product.

Method 1. The cross product of two vectors in $\mathbb{R}^3$ can be determined by the following information.

(i) $\vec{i} \times \vec{j} = \vec{k}$, $\vec{j} \times \vec{k} = \vec{i}$, and $\vec{k} \times \vec{i} = \vec{j}$ (The first picture on the next page shows this multiplication table.)

(ii) The cross product of any vector with itself is zero.

(iii) Cross product is anti-commutative; in other words,

$$\vec{v} \times \vec{w} = -\vec{w} \times \vec{v},$$

for all vectors $\vec{v}$ and $\vec{w}$.

(iv) Cross product distributes over addition.

Method 2. The cross product $\vec{v} \times \vec{w}$ is a vector. The length of $\vec{v} \times \vec{w}$ is equal to the area of the parallelogram determined by $\vec{v}$ and $\vec{w}$. (The second picture on the next page shows the parallelogram determined by $\vec{v}$ and $\vec{w}$.) The cross product $\vec{v} \times \vec{w}$ is perpendicular to both $\vec{v}$ and $\vec{w}$ and is obtained using the right hand rule. The right hand rule says "place your right hand in the plane of $\vec{v}$ and $\vec{w}$ with your fingers curling from $\vec{v}$ to $\vec{w}$, then your thumb points in the direction of $\vec{v} \times \vec{w}$." (The third picture on the next page shows the right hand rule.)

Method 3. If $\vec{a} = a_1\vec{i} + a_2\vec{j} + a_3\vec{k}$ and $\vec{b} = b_1\vec{i} + b_2\vec{j} + b_3\vec{k}$, then

$$\vec{a} \times \vec{b} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = \begin{vmatrix} a_2 & a_3 \\ b_2 & b_3 \end{vmatrix} \vec{i} - \begin{vmatrix} a_1 & a_3 \\ b_1 & b_3 \end{vmatrix} \vec{j} + \begin{vmatrix} a_1 & a_2 \\ b_1 & b_2 \end{vmatrix} \vec{k}.$$

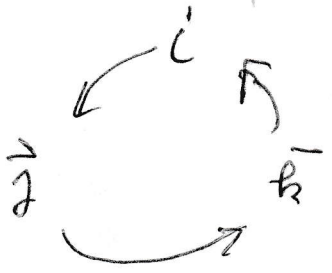
The rectangular arrays are called matrices. The big vertical lines tell us to take the determinant of the matrix. I took the determinant by "expanding across the first row". The determinant of the 3×3 matrix is equal to the entry in row 1 column 1 times the determinant of the matrix obtained by deleting row 1 and column 1 of the big matrix **minus**[1] the entry in row 1 column 2 times the determinant of the matrix obtained by deleting row 1 and column 2 of the big matrix plus the entry in row 1 column 3 times the determinant of the matrix obtained by deleting row 1 and column 3 of the big matrix. The determinant of the 2×2 matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$$

is $ad - bc$.

[1]This minus is very important. If you have computed a cross product and it does not seem to be working correctly, then there is a very good chance that you forgot this minus sign.

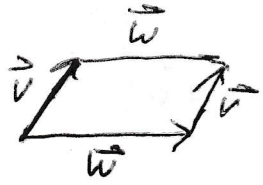
The $\vec{i}, \vec{j}, \vec{k}$ Picture



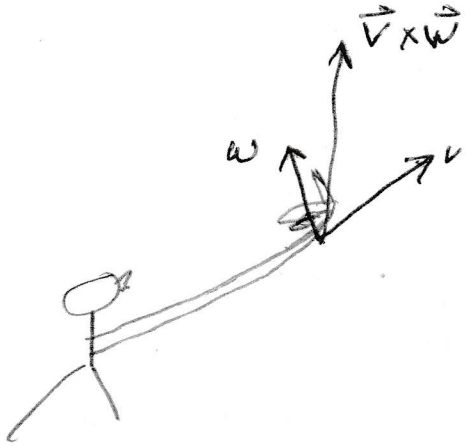
$\vec{i} \times \vec{j} = \vec{k}$

$\vec{j} \times \vec{k} = \vec{i}$

$\vec{k} \times \vec{i} = \vec{j}$



The parallelogram determined by $\vec{v}$ and $\vec{w}$



The right hand rule. Put your right hand in the plane of $\vec{v}$ and $\vec{w}$ with your fingers curling from $\vec{v}$ to $\vec{w}$.

Your thumb points in the direction of $\vec{v} \times \vec{w}$.

Examples 5.1. (a) Notice that $\vec{i}$ and $\vec{j}$ determine a parallelogram of area one; that the right hand rule shows that $\vec{i} \times \vec{j}$ points in the direction of $\vec{k}$; and that the right hand rule shows that $\vec{j} \times \vec{i}$ points in the direction of $-\vec{k}$.

(b) Notice that $\vec{j}$ and $\vec{k}$ determine a parallelogram of area one; that the right hand rule shows that $\vec{j} \times \vec{k}$ points in the direction of $\vec{i}$; and that the right hand rule shows that $\vec{k} \times \vec{j}$ points in the direction of $-\vec{i}$.

(c) Notice that $\vec{k}$ and $\vec{i}$ determine a parallelogram of area one; that the right hand rule shows that $\vec{k} \times \vec{i}$ points in the direction of $\vec{j}$; and that the right hand rule shows that $\vec{i} \times \vec{k}$ points in the direction of $-\vec{j}$.

(d) Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = -\vec{i} + 2\vec{j}$. Compute $\vec{a} \times \vec{b}$ (**and verify that your answer is perpendicular to $\vec{a}$ and $\vec{b}$**).

$$\begin{aligned}\vec{a} \times \vec{b} &= \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 2 & 3 \\ -1 & 2 & 0 \end{vmatrix} \\ &= \begin{vmatrix} 2 & 3 \\ 2 & 0 \end{vmatrix} \vec{i} - \begin{vmatrix} 1 & 3 \\ -1 & 0 \end{vmatrix} \vec{j} + \begin{vmatrix} 1 & 2 \\ -1 & 2 \end{vmatrix} \vec{k} \\ &= \boxed{-6\vec{i} - 3\vec{j} + 4\vec{k}}.\end{aligned}$$

Check. Observe that $\vec{a} \cdot$ our answer is

$$(\vec{i} + 2\vec{j} + 3\vec{k}) \cdot (-6\vec{i} - 3\vec{j} + 4\vec{k}) = -6 - 6 + 12 = 0\checkmark$$

and $\vec{b} \cdot$ our answer is

$$(-\vec{i} + 2\vec{j}) \cdot (-6\vec{i} - 3\vec{j} + 4\vec{k}) = 6 - 6 = 0\checkmark.$$

(e) Let $\vec{a} = \vec{i} + 2\vec{j} + 3\vec{k}$ and $\vec{b} = -\vec{i} + 2\vec{j}$. Use Method 1 to calculate $\vec{a} \times \vec{b}$. We see that

$$\begin{aligned}\vec{a} \times \vec{b} &= (\vec{i} + 2\vec{j} + 3\vec{k}) \times (-\vec{i} + 2\vec{j}) \\ &= (\vec{i} + 2\vec{j} + 3\vec{k}) \times (-\vec{i}) + (\vec{i} + 2\vec{j} + 3\vec{k}) \times (2\vec{j}) \\ &= \begin{cases} -(\vec{i} \times \vec{i}) - 2(\vec{j} \times \vec{i}) - 3(\vec{k} \times \vec{i}) \\ +2(\vec{i} \times \vec{j}) + 4(\vec{j} \times \vec{j}) + 6(\vec{k} \times \vec{j}) \end{cases} \\ &= -\vec{0} - 2(-\vec{k}) - 3\vec{j} + 2\vec{k} + 4(\vec{0}) + 6(-\vec{i}) \\ &= -6\vec{i} - 3\vec{j} + 4\vec{k}.\end{aligned}$$

Of course, this is the same answer as the one we got in (d). In practice, most of the time, one uses Method 3 to compute cross product. It merely provides an organized way to compute cross product using the rules of Method 1.

6. Section 12.5. Lines and planes.

Section 12.5 contains a large amount of material.

A. We find the equation of a plane.
B. We find the equations of a line.
C. Given a point P and a plane. We find the point on the plane which is closest to P.
D. Given a point P and a line ℓ. We find the point on ℓ closest to P.

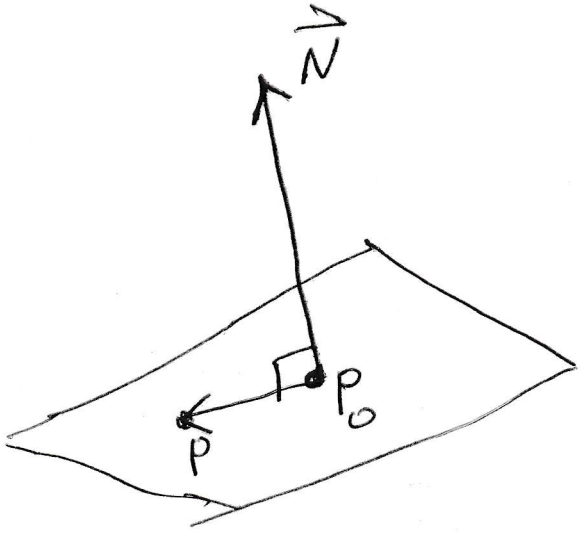
6.A. The equation of a plane.

What geometric data describes a plane?

One plausible answer is three point determine a plane (unless the 3 points happen to live on a line). This is a very good answer, except there is a more efficient answer. This data requires nine numbers. (We have to give three coordinates for each point.) If this is the best we could do, then we would have a formula with nine constants in it. Ugh!

A second answer is if P_0 is an arbitrary point and $\overrightarrow{N}$ is an arbitrary vector, then there is exactly one plane that contains P_0 and is perpendicular to $\overrightarrow{N}$. See the picture on the next page.

The plane through P_0 and perpendicular to $\vec{N}$



If P is an arbitrary point in 3-space, then P is on the plane through P_0 perpendicular to $\vec{N}$ if and only if

$$\overrightarrow{P_0P} \perp \vec{N}.$$

We conclude that the plane through $P_0 = (x_0, y_0, z_0)$ perpendicular to the vector $\overrightarrow{N} = A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k}$ is the set of all points $P = (x, y, z)$ with

$$\overrightarrow{P_0P} \perp \overrightarrow{N}.$$

In other words, $A(x-x_0)+B(y-y_0)+c(z-z_0) = 0$.

Conclusion 6.1. *The equation of the plane through $P_0 = (x_0, y_0, z_0)$ perpendicular to the vector $\overrightarrow{N} = A\overrightarrow{i} + B\overrightarrow{j} + C\overrightarrow{k}$ is $A(x-x_0)+B(y-y_0)+c(z-z_0) = 0$.*

Question 6.2. *Find the equation of the plane through $P_0 = (1,2,3)$ perpendicular to $\overrightarrow{N} = 2\overrightarrow{i} - 3\overrightarrow{j} + 5\overrightarrow{k}$.*

Answer. Use Conclusion 6.1 to see that the answer is

$$\boxed{2(x-1)-3(y-2)+5(z-3) = 0.}$$

Question 6.3. *What vector is perpendicular to the plane $6x - 9y + 10z = 28$?*

Answer. Use Conclusion 6.1 to see that the answer is

$$\boxed{6\overrightarrow{i} - 9\overrightarrow{j} + 10\overrightarrow{k}.}$$

Question 6.4. *Find the equation of the plane through $P_1 = (2,3,4)$, $P_2 = (3,4,5)$, and $P_3 = (1,6,8)$.*

Answer. The vectors $\overrightarrow{P_1P_2}$ and $\overrightarrow{P_1P_3}$ both lie on the plane we are supposed to find. Thus, $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the plane we are supposed to find. (Look at the picture on the next page, if necessary.) We compute

$$\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3} = \begin{vmatrix} \overrightarrow{i} & \overrightarrow{j} & \overrightarrow{k} \\ 1 & 1 & 1 \\ -1 & 3 & 4 \end{vmatrix} = \begin{vmatrix} 1 & 1 \\ 3 & 4 \end{vmatrix}\overrightarrow{i} - \begin{vmatrix} 1 & 1 \\ -1 & 4 \end{vmatrix}\overrightarrow{j} + \begin{vmatrix} 1 & 1 \\ -1 & 3 \end{vmatrix}\overrightarrow{k}$$

$$= \overrightarrow{i} - 5\overrightarrow{j} + 4\overrightarrow{k}.$$

The plane through $P_1 = (2,3,4)$ perpendicular $\overrightarrow{i} - 5\overrightarrow{j} + 4\overrightarrow{k}$ is

$$(x-2)-5(y-3)+4(z-4) = 0.$$

$$\boxed{x - 5y + 4z = 3.}$$

CHECK: Plug $P_1 = (2,3,4)$ into the proposed answer:

$$2 - 15 + 16 = 3\checkmark$$

Plug $P_2 = (3,4,5)$ into the proposed answer:

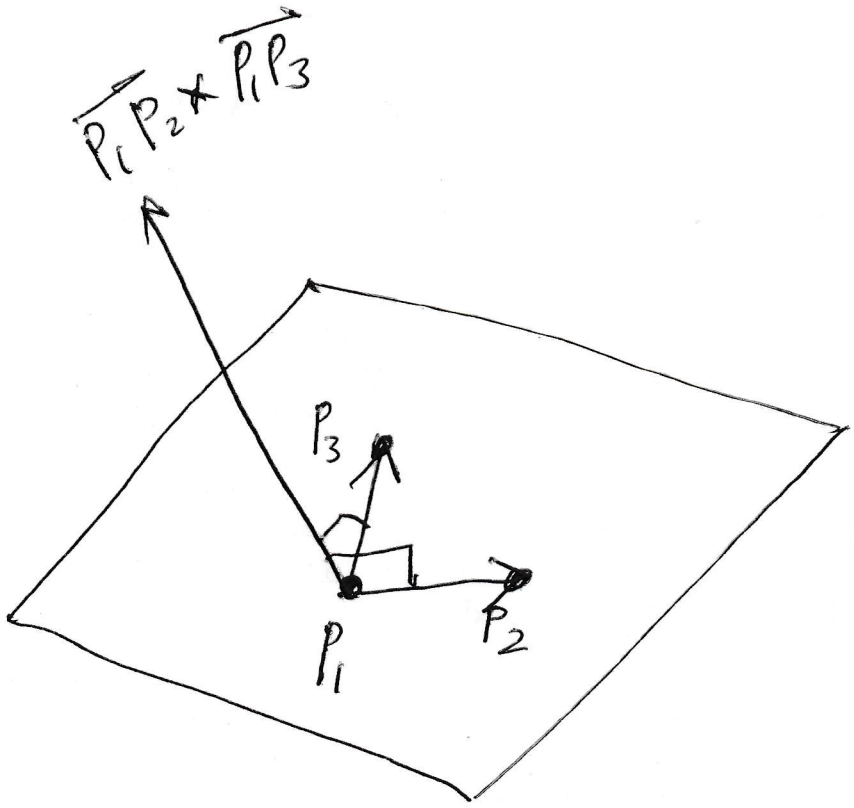
$$3 - 20 + 20 = 3\checkmark$$

Plug $P_3 = (1,6,8)$ into the proposed answer:

$$1 - 30 + 32 = 3\checkmark$$

3 Points on a Plane.

If P_1, P_2, P_3 lie on a plane, then $\overrightarrow{P_1P_2} \times \overrightarrow{P_1P_3}$ is perpendicular to the plane



6.B. **The equations of a line.**

Question. *What geometric data describes a line?*

Answer. 1. One correct answer is "Two points determine a line".
2. Another correct answer is "A point on the line and a vector parallel to the line determine a line."

We use answer the second set of data. (Of course, the first set of data can be easily transformed into the second set of data.)

Question. *What are the parametric equations for the line through* $P_0 = (x_0, y_0, z_0)$ *and parallel to* $\overrightarrow{\boldsymbol{v}} = a\overrightarrow{\boldsymbol{i}} + b\overrightarrow{\boldsymbol{j}} + c\overrightarrow{\boldsymbol{k}}$ *?*

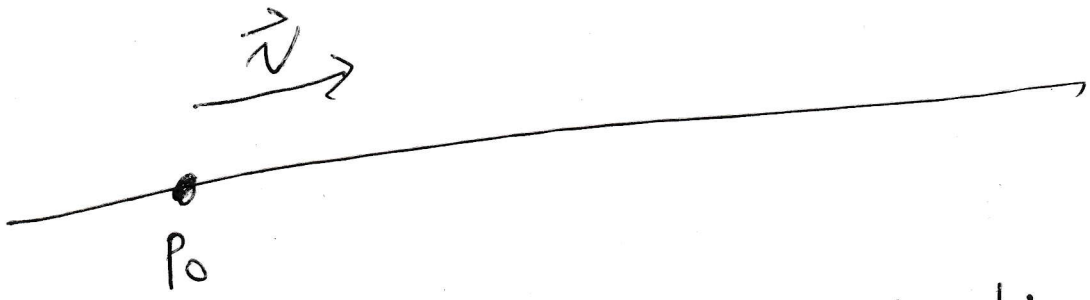
Answer. Look at the picture! The point $P = (x, y, z)$ is on the line if and only if

$$\overrightarrow{P_0P} = t\overrightarrow{\boldsymbol{v}}$$

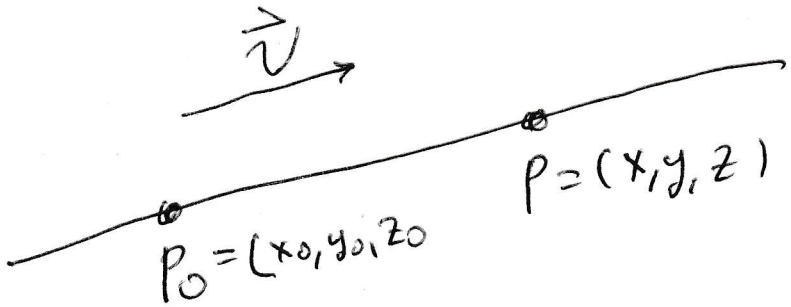
for some number t.

The equations of a line.

Consider the line through the point $P_0 = (x_0, y_0, z_0)$ and parallel to the vector $\vec{v} = a\vec{i} + b\vec{j} + c\vec{k}$.



The point $P = (x, y, z)$ is on the line if and only if $\overrightarrow{P_0P} = t\vec{v}$ for some number t.



The point $P=(x,y,z)$ is on the line if and only if

$$\overrightarrow{P_0P} = t\,\overrightarrow{\boldsymbol{v}}$$

for some number t.

The point $P=(x,y,z)$ is on the line if and only if

$$(x-x_0)\,\overrightarrow{\boldsymbol{i}} + (y-y_0)\,\overrightarrow{\boldsymbol{j}} + (z-z_0)\,\overrightarrow{\boldsymbol{k}} = t(a\,\overrightarrow{\boldsymbol{i}} + b\,\overrightarrow{\boldsymbol{j}} + c\,\overrightarrow{\boldsymbol{k}})$$

for some number t.

Fact 6.5. *The line through the point $P_0=(x_0,y_0,z_0)$ and parallel to the vector $v = a\,\overrightarrow{\boldsymbol{i}} + b\,\overrightarrow{\boldsymbol{j}} + c\,\overrightarrow{\boldsymbol{k}}$ is*

$$\begin{cases} x = x_0 + at \\ y = y_0 + bt \\ z = z_0 + ct. \end{cases}$$

Example. Find parametric equations for the line through $P_0=(1,0,0)$ parallel to the vector $v = \overrightarrow{\boldsymbol{i}} + 2\,\overrightarrow{\boldsymbol{j}}$.

This is the line

$$\begin{cases} x = 1 + t \\ y = 0 + 2t \\ z = 0. \end{cases}$$

There is a picture on the next page. (This particular line lives on the xy-plane. The parallel vector $\overrightarrow{\boldsymbol{i}} + 2\,\overrightarrow{\boldsymbol{j}}$ conveys the information that the line has slope 2.)

Question. *Find parametric equations for the line through $P=(1,2,3)$ and $Q=(8,9,11)$.*

Answer. The line through $P=(1,2,3)$ which is parallel to

$$\overrightarrow{PQ} = \overrightarrow{(1,2,3)(8,9,11)} = 7\,\overrightarrow{\boldsymbol{i}} + 7\,\overrightarrow{\boldsymbol{j}} + 8\,\overrightarrow{\boldsymbol{k}}$$

is

$$\boxed{\begin{cases} x = 1 + 7t \\ y = 2 + 7t \\ z = 3 + 8t. \end{cases}}$$

Check: If one walks on the line, then at $t=0$ one is standing on $(1,2,3)$ and at $t=1$ one is standing on $(8,9,11)$.

Question. *Give a vector that is parallel to the line*

$$\begin{cases} x = 6 + 4t \\ y = 11 - 7t \\ z = 12 + t \end{cases}.$$

Answer. The vector $\overrightarrow{v} = 4\overrightarrow{i} - 7\overrightarrow{j} + \overrightarrow{k}$ is parallel to the line

$$\begin{cases} x = 6+4t \\ y = 11-7t \\ z = 12+t \end{cases}.$$

6.C. **Find the point on the plane $Ax+By+Cz=D$ which is closest to the point $P_1 = (x_1, y_1, z_1)$.**

Example. 6.8 Find the point Q_1 on $x+3y+2z=4$ which is closest to the point $P_1 = (1,2,3)$ and **check your answer.**

Step 1. We find the line through $P_1 = (1,2,3)$ which is parallel to $\overrightarrow{N} = \overrightarrow{i} + 3\overrightarrow{j} + 2\overrightarrow{k}$

Step 2. The point Q_1 is the intersection of the line from Step 1 and the plane.

We carry out Step 1. The line is

$$\begin{cases} x = 1+t \\ y = 2+3t \\ z = 3+2t \end{cases}$$

We carry out Step 2. We first find when the line and the plane intersect. (In my mind, something is walking along the line. The parametric equations for the line tell where the object is at each moment. We first find the moment that puts the object on the plane.) The object is on the plane when

$$(1+t)+3(2+3t)+2(3+2t) = 4.$$

This simplifies to

$$1+6+6+(1+9+4)t = 4$$
$$14t = -9$$
$$t = \frac{-9}{14}$$

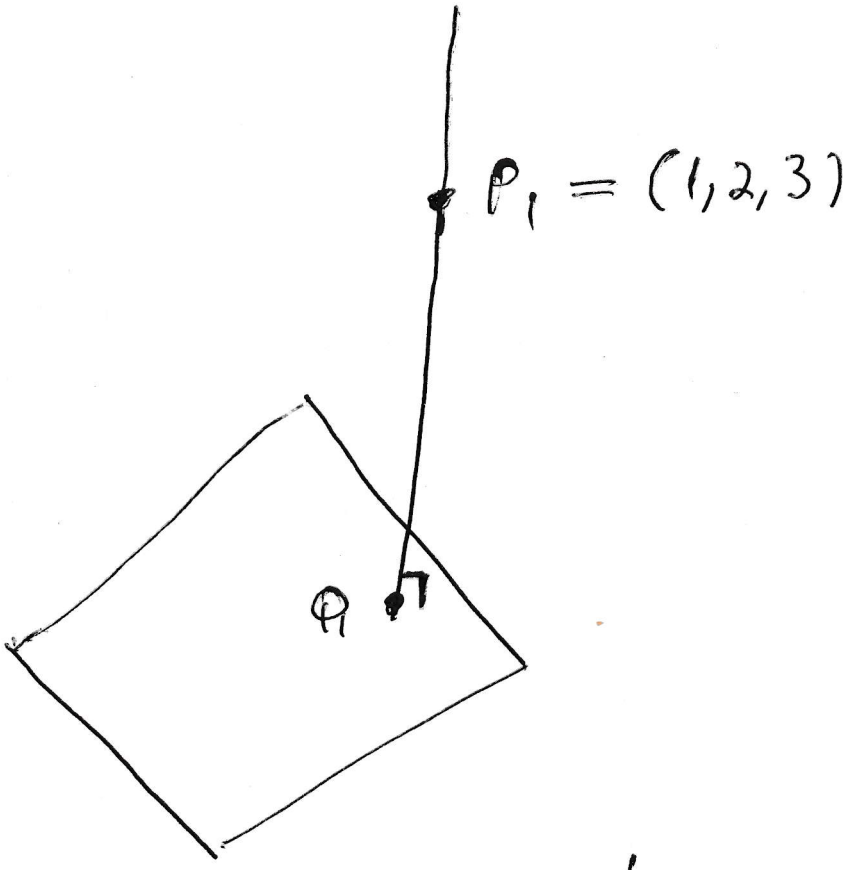
At time $t = \frac{-9}{14}$ the object stands on the plane and the position of the object is

$$\begin{cases} x = 1+\frac{-9}{14} = \frac{5}{14} \\ y = 2-\frac{27}{14} = \frac{1}{14} \\ z = 3-\frac{18}{14} = \frac{24}{14} \end{cases}$$

So

$$\boxed{Q_1 = \left(\frac{5}{14}, \frac{1}{14}, \frac{24}{14}\right).}$$

The Picture for Example 6.8



The plane is $X+3Y+2Z=4$. The line is $\begin{cases} X = 1+t \\ Y = 2+3t \\ Z = 3+2t \end{cases}$

The line and the plane meet when $t = \frac{-9}{14}$.

We verify that Q_1 is on the plane by plugging $\left(\frac{5}{14}, \frac{1}{14}, \frac{24}{14}\right)$ into $x+3y+2z=4$:

$$\frac{5}{14}+3\left(\frac{1}{14}\right)+2\left(\frac{24}{14}\right)=\frac{56}{14}=4.\checkmark$$

We verify that $\overrightarrow{Q_1P_1}$ is perpenicular to the plane. Well,

$$\overrightarrow{Q_1P_1}=\overrightarrow{\left(\frac{5}{14},\frac{1}{14},\frac{24}{14}\right)\left(\frac{14}{14},\frac{28}{14},\frac{42}{14}\right)}=\frac{9}{14}\vec{i}+\frac{27}{14}\vec{j}+\frac{18}{14}\vec{k}=\frac{9}{14}(\vec{i}+3\vec{j}+2\vec{k})$$

which is perpendicular yo the plane, as expected.

6.D. **Given a line ℓ and a point P_1 find the point on ℓ closest to P_1.**

Example. Find the point on the line

$$\ell=\begin{cases}x=2t\\ y=1+2t\\ z=2t\end{cases}$$

which is closest to the point $P_1=(1,2,3)$.[2]

Step 1. We find the plane through $P_1=(1,2,3)$ which is perpendicular to the line

$$\begin{cases}x=2t\\ y=1+2t\\ z=2t.\end{cases}$$

Step 2. The answer is the intersection of the original line and the plane of Step 1. (See the picture at the end of the problem.)

We carry out Step 1. The given line is parallel to $\vec{v}=2\vec{i}+2\vec{j}+2\vec{k}$; is the plane that we want to create is perpendicular to $\vec{v}$ and passes through $P_1=(1,2,3)$. Our plane is $2(x-1)+2(y-2)+2(z-3)=0$, which can be rewritten as $x+y+z=6$.

We carry out Step 2. The line

$$\begin{cases}x=2t\\ y=1+2t\\ z=2t\end{cases}$$

and the plane $x+y+z=12$ meet, when

$$2t+(1+2t)+2t=6$$

$$6t=5$$

$$t=\frac{5}{6}.$$

[2]We did a different point in class. Both problems (the one done here and the one done in class) are done correctly.

If an object moves along the line, then at time $t = \frac{5}{6}$ the position of the object is

$$x = \frac{5}{3}, \quad y = \frac{8}{3}, \quad \text{and} \quad z = \frac{5}{3}.$$

The point on

$$\begin{cases} x = 2t \\ y = 1 + 2t \\ z = 2t. \end{cases}$$

closest to $P_1 = (1,2,3)$ is

$$\boxed{\left(\frac{5}{3}, \frac{8}{3}, \frac{5}{3}\right)}$$

Check. We see that $\left(\frac{5}{3}, \frac{8}{3}, \frac{5}{3}\right)$ is on the line

$$\begin{cases} x = 2t \\ y = 1 + 2t \\ z = 2t \end{cases}$$

when $t = \frac{5}{6}$. We also see that the vector

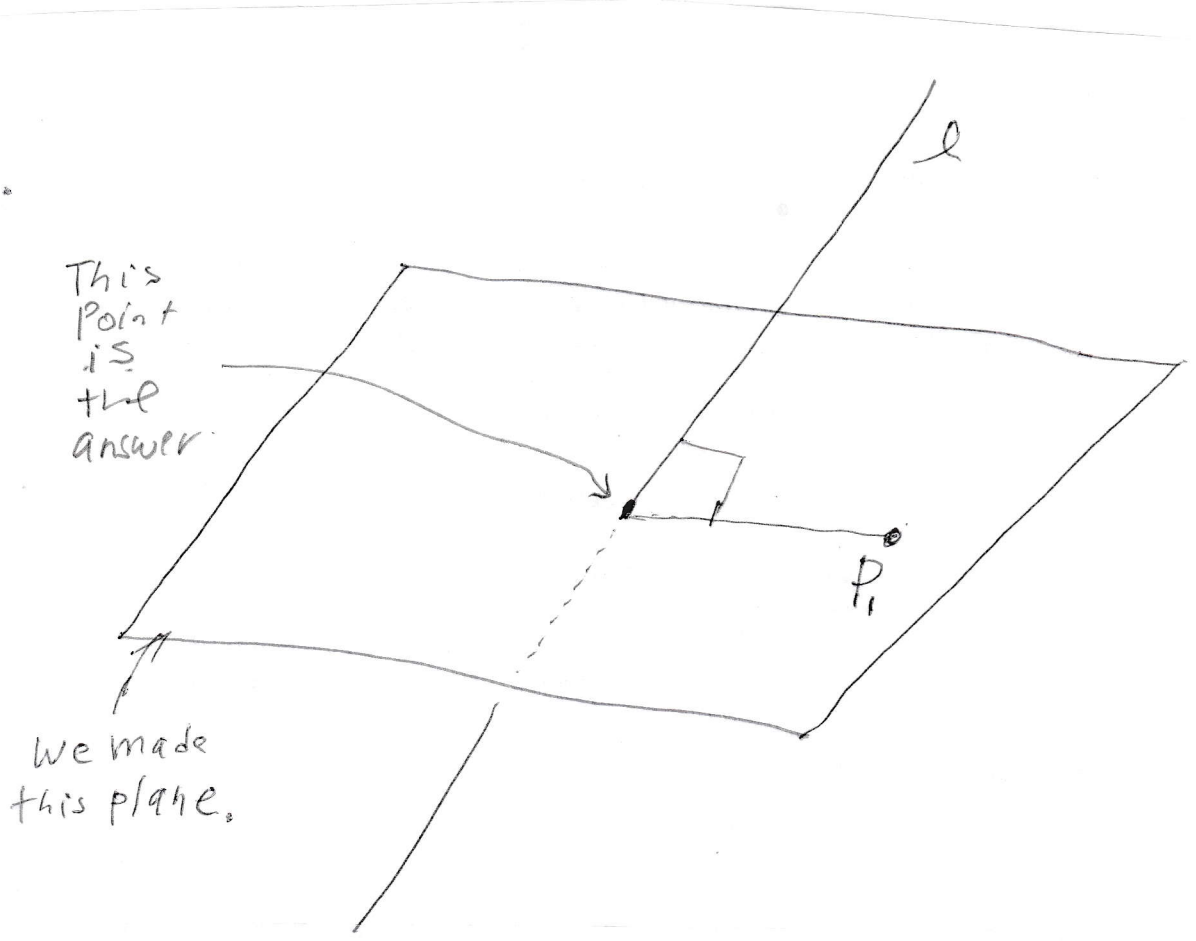
$$\overrightarrow{(1,2,3)\left(\frac{5}{3}, \frac{8}{3}, \frac{5}{3}\right)} = \frac{2}{3}\vec{\boldsymbol{i}} + \frac{2}{3}\vec{\boldsymbol{j}} - \frac{4}{3}\vec{\boldsymbol{k}}$$

is perpendicular to the line because

$$(\tfrac{2}{3}\vec{\boldsymbol{i}} + \tfrac{2}{3}\vec{\boldsymbol{j}} - \tfrac{4}{3}\vec{\boldsymbol{k}}) \cdot (2\vec{\boldsymbol{i}} + 2\vec{\boldsymbol{j}} + 2\vec{\boldsymbol{k}}) = 0.$$

Given a Point P_1 and a line ℓ, find the point on ℓ nearest P_1

First find the equation of Plane through P_1 Perpendicular to ℓ. the answer is the intersection of ℓ and the plane



7. Section 12.6. Surfaces in three space.

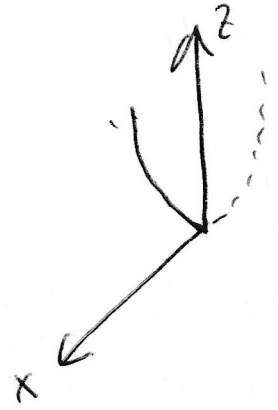
We focus on two situations.

A. A variable is missing.
B. The equation has degree two.

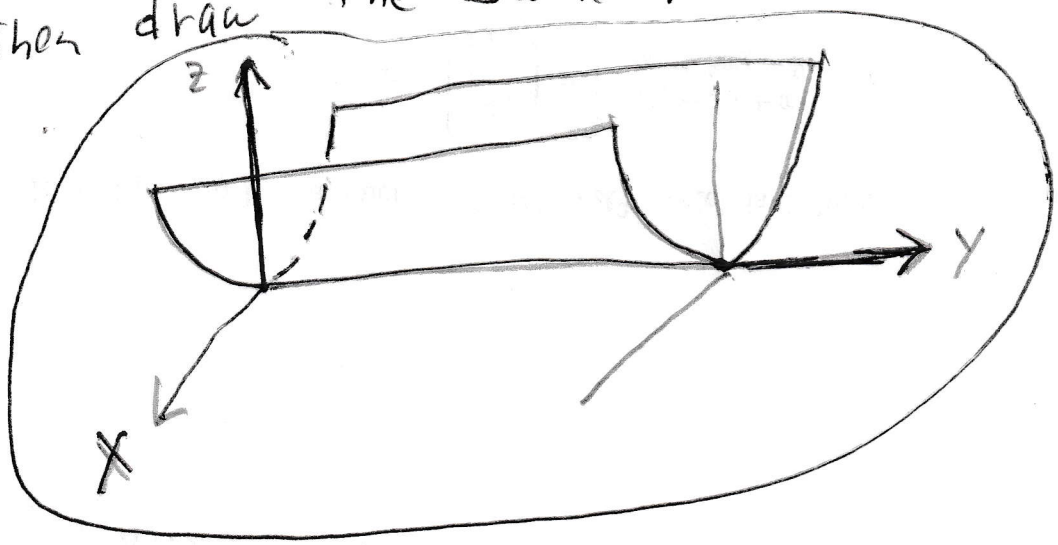
7.A. A variable is missing.

Example 7.1. To graph $z = x^2$ in three-space. Graph $z = x^2$ in the xz-plane. Then graph the same picture for each y. The graph is called a cylinder. (If we want to emphasize that the cross sections are parabolas we call it a parabolic cylinder.) The picture is on the next page.

To graph $z = x^2$ in 3-space, draw $z = x^2$ on the xz plane



Then draw the same picture for each y



The graph is called a parabolic cylinder.

7.B. **The equation has degree two.** To graph a quadric surface (that is the equation is quadratic), see what the graph looks like when $x = 0$, $y = 0$, $z = 0$, and maybe x equals constant, y equals constant, z equals constant, then connect the pieces. (When you set $x = 0$ you are taking the cross section of the surface which lives in the yz plane. The hope is that you can figure out what the surface looks like by carefully studying a few cross sections.)

Example 7.2. Graph $x^2 + 4y^2 = z^2$ in three space.

When $x = 0$ the equation becomes $4y^2 = z^2$ and the graph is two lines. One of the lines is $2y = z$ and the other line is $-2y = z$.

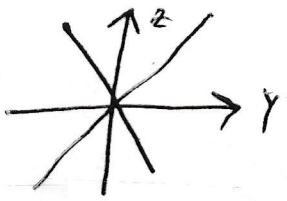
When $y = 0$ the graph is two lines; namely $x = z$ together with $y = z$.

When $z = 0$ the equation is $x^2 + 4y^2 = 0$. The number x^2 is zero or higher. The number $4y^2$ is zero or higher. If two numbers, each of which is zero or higher, add to zero, then both numbers are zero. So the cross section of the surface $x^2 + 4y^2 = z^2$ that lives in the xy-plane is just the origin. This is legal, but not very informative. Maybe we should look at the cross section of the surface when z is the non-zero constant c. Well, $x^2 + 4y^2 = c^2$ is an ellipse. (If the 4 weren't there then the equation would be a circle. An ellipse is a circle that is stretched a little in one direction.

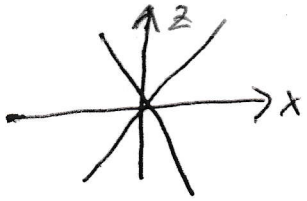
The rest of the story is on the next page.

Graph $X^2 + 4Y^2 = Z^2$ in 3 space

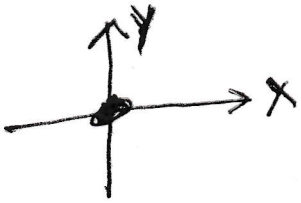
when $X = 0$



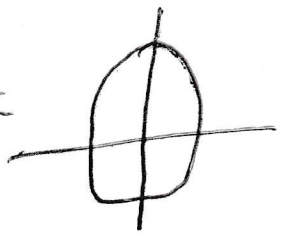
when $Y = 0$

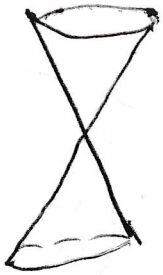


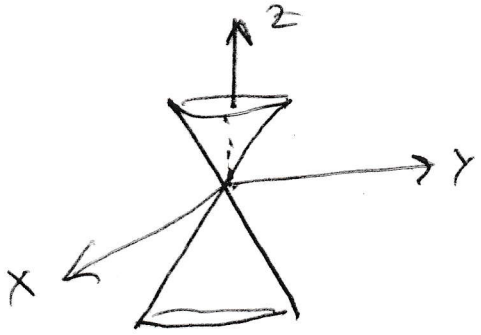
when $Z = 0$



when $Z = c$







The graph is a cone.

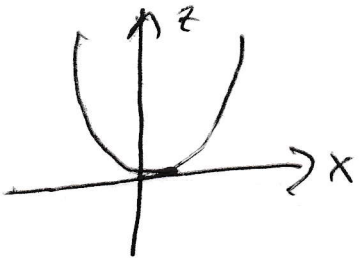
(I can usually draw the picture better without the axes and then I put the axes in later.)

Example 7.3. Graph $z = x^2 + y^2$ in three space. The cross section when $y = 0$ is the parabola $z = x^2$. The cross section when $x = 0$ is the parabola $z = y^2$. The cross section when $z = 0$ **is a point**. (Again, x^2 and y^2 are both zero or higher. If x^2 plus y^2 equals zero, then x and y are both zero.) If z is a positive constant c, then $c = x^2 + y^2$ is a circle. There is no graph when z is negative.

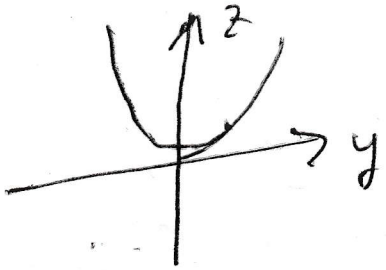
The rest of the story is on the next page.

Graph $z = x^2 + y^2$ in three space.

When $y=0$, the graph is a parabola:

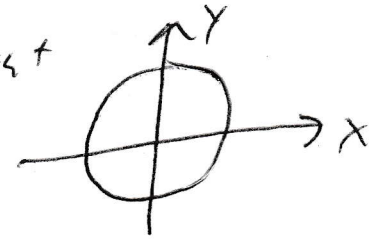


when $x=0$, the graph is a parabola



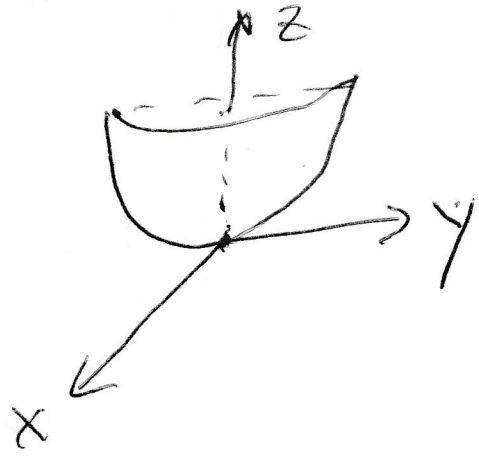
when $z=0$, the graph is the origin.

If z is a positive constant the graph is a circle



The graph is

or



The graph is called a paraboloid.

Example 7.4. Graph $z^2 + x^2 - y^2 = 1$ in three space.

The cross section when $x = 0$ is the hyperbola $z^2 - y^2 = 1$. (Notice that $(y,z) = (0,1)$ is on the graph, but when $z = 0$, there is no graph.)

The cross section when $z = 0$ is the hyperbola $x^2 - y^2 = 1$. (Notice that $(x,y) = (1,0)$ is on the graph, but when $x = 0$, there is no graph.)

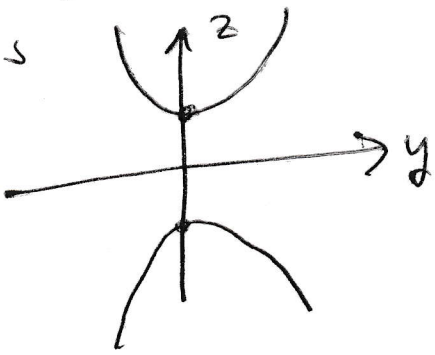
The cross section when $y = 0$ is the circle $z^2 + x^2 = 1$.

The rest of the story is on the next page.

The surface in three space is called a "hyperboloid of one sheet". The phrase "one sheet" is an old fashioned way of saying that the surface consists of one piece. A little later we will study a hyperboloid which comes in two pieces, we call it a hyperboloid of two sheets.

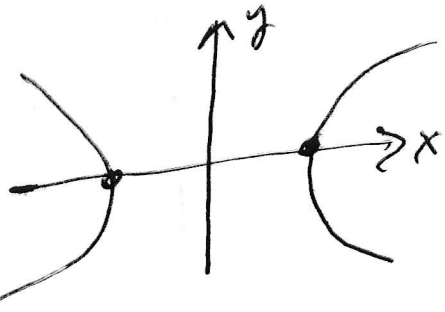
Graph $z^2 + x^2 - y^2 = 1$ in three space

When $x = 0$ the equation is $z^2 - y^2 = 1$ and the graph is



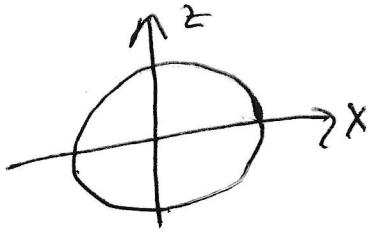
a hyperbola

When $z = 0$ the equation is $x^2 - y^2 = 1$ and the graph is

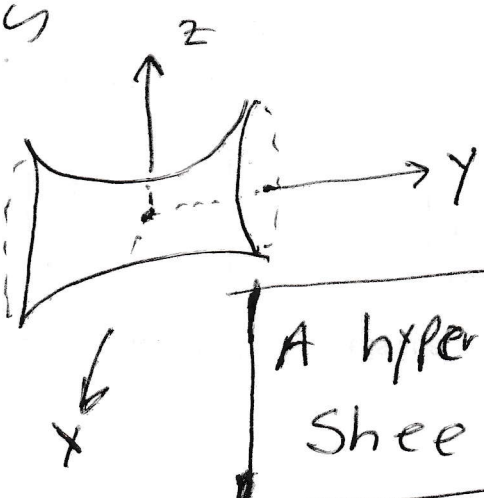


a hyperbola

When $y = 0$ the equation is $z^2 + x^2 = 1$ a circle



The surface is



A hyperboloid of one sheet.

Example 7.5. Graph $z = y^2 - x^2$ in three space.

The cross section when $x = 0$ is the parabola $z = y^2$.

The cross section when $y = 0$ is the parabola $z = -x^2$.

The cross section when $z = 0$ is $0 = y^2 - x^2$. This is two lines.[3] Everything on $y = x$ satisfies $0 = y^2 - x^2$. Also everything on $y = -x$ satisfies $0 = y^2 - x^2$.

At any rate two lines is probably not a good representative of the cross sections in this direction. If z is a positive constant c, then $c = y^2 - x^2$ is a hyperbola. In this hyperbola, (x, y) can be $(0, \pm\sqrt{c})$, but y can not be zero. If z is a negative constant, then $c = y^2 - x^2$ or $|c| = x^2 - y^2$, which is also a hyperbola. This hyperbola contains the points $(x, y) = (\pm\sqrt{|c|}, 0)$, but x can not be zero.

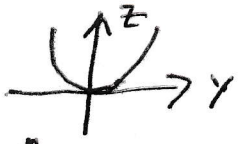
The resulting surface drawn on the next page. It is called a hyperbolic paraboloid or a "saddle surface". This is an important surface for us because the origin turns out to be a "critical point" (when we are applying the techniques of section 14.7 to find local extreme points). If we study the cross section of the surface that is in the yz-plane, then we would say that the origin is certainly a local minimum. However if we focus on the cross section in the xz plane, then we would say that the origin is certainly a local maximum. It turns out that the origin is neither a local minimum or a local maximum of this surface. The origin is called a saddle point of the surface.

The rest of the story is on the next page.

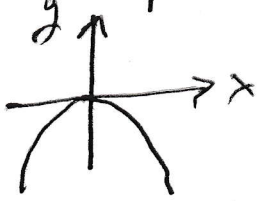
[3] Maybe you want to move x^2 to the other side (so $x^2 = y^2$) and then take the square root of both sides (so $x = y$ or $x = -y$). The other way to look at $0 = y^2 - x^2$ is to factor $0 = (y - x)(y + x)$. If a product of two numbers is zero, then at least one of the two numbers is zero. Again we obtain $x = y$ or $x = -y$.

Graph $z = y^2 - x^2$ in Three space

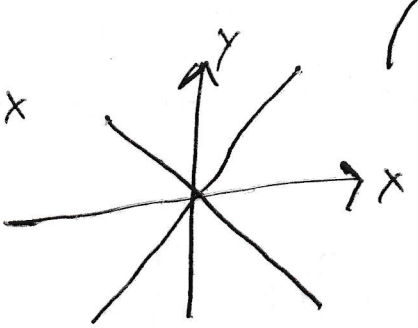
When $x = 0$, $z = y^2$ is a parabola



When $y = 0$, $z = -x^2$ is a parbola

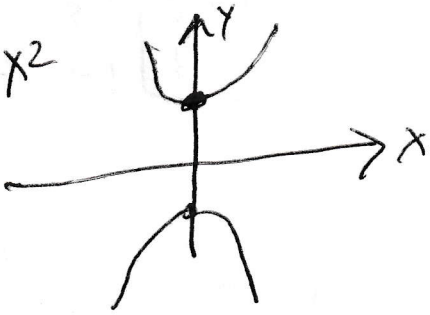


when $z = 0$ $y = x$ or $y = -x$



When z = Positive constant c

$c = y^2 - x^2$



hyperbola

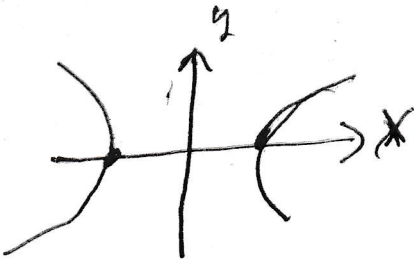
When z = negative constant c

$c = y^2 - x^2$

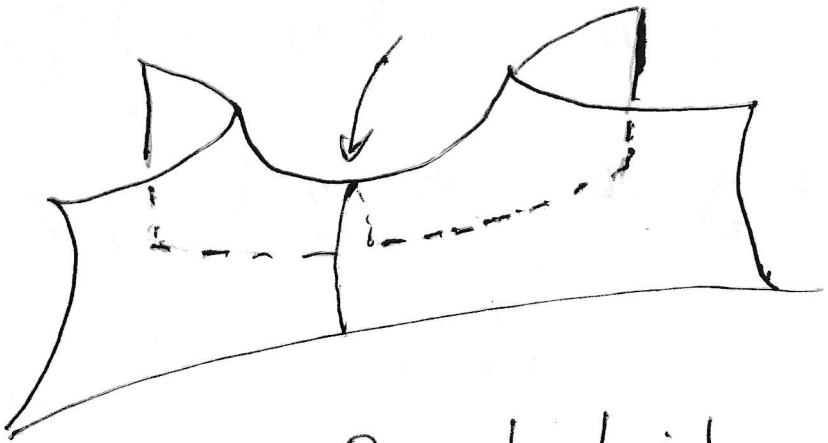
$\therefore -c = x^2 - y^2$

$|c| = x^2 - y^2$

but $-c = |c|$



Saddle Point



Hyperbolic Paraboloid

Example 7.6. Graph $z^2 - x^2 - y^2 = 1$ in three space.

The cross section when $x = 0$ is the hyperbola $z^2 - y^2 = 1$. (Notice that $(y,z) = (0,1)$ is on the graph, but z can not be zero.)

The cross section when $y = 0$ is the hyperbola $z^2 - x^2 = 1$. (Notice that $(x,z) = (0,1)$ is on the graph, but z can not be zero.)

When $z = 0$, there is no graph. Indeed, there is no graph for $-1 < z < 1$.

When $z = 1$ or $z = -1$, the graph is a point.

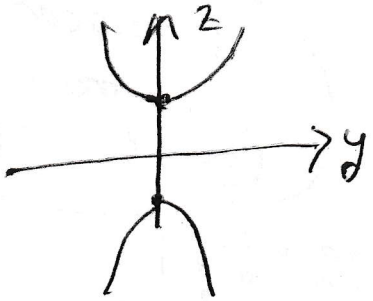
When z is a constant c with $1 < c$ or $c < -1$, then the graph is the circle $c^2 - x^2 - y^2 = 1$, which is the same as $c^2 - 1 = x^2 + y^2$.

The graph of $z^2 - x^2 - y^2 = 1$ in three space is called "a hyperboloid of two sheets". "Of two sheets is an old fashioned way of saying, "in two pieces".

Please see the next page.

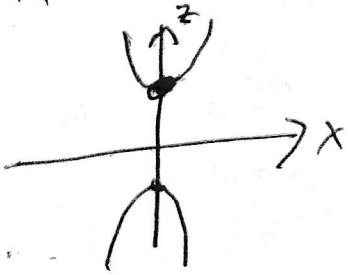
Graph $z^2 - x^2 - y^2 = 1$ in three space

The cross section when $x = 0$ is $z^2 - y^2 = 1$



hyperbola

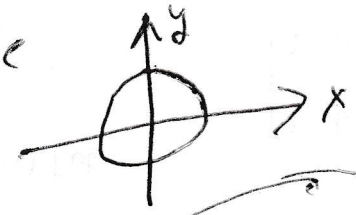
The cross section when $y = 0$ is $z^2 - x^2 = 1$

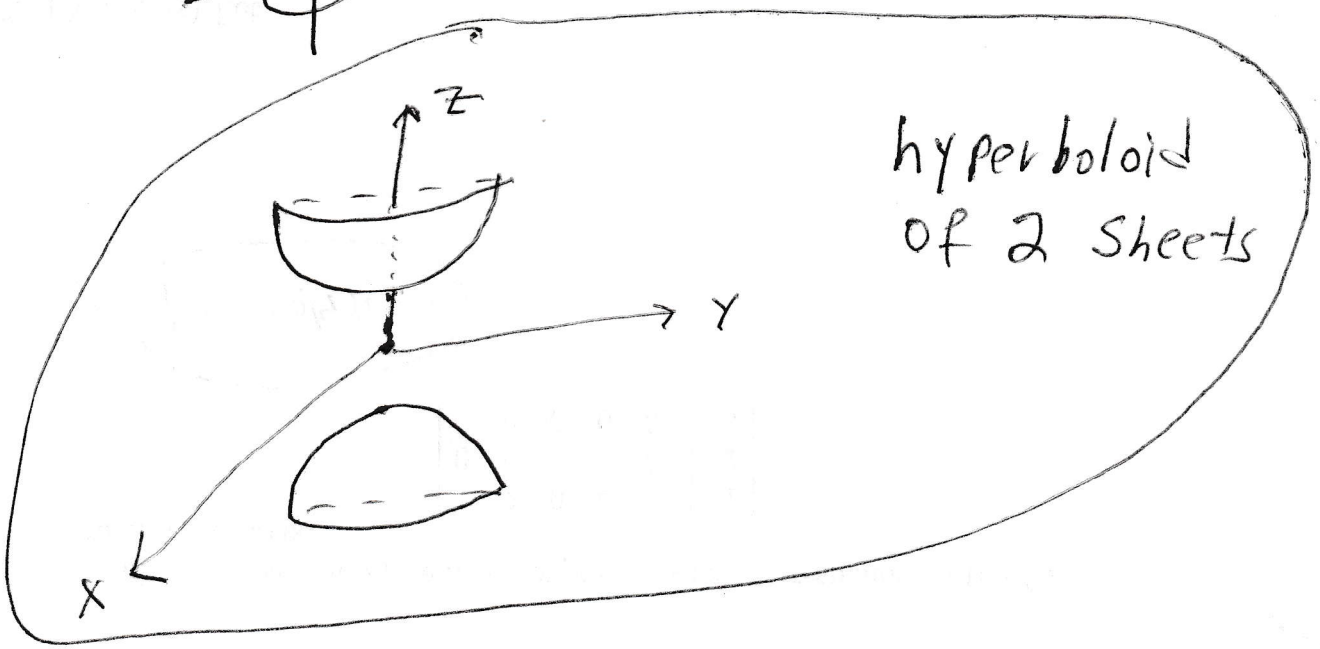


hyperbola

When $-1 < z < 1$, there is no graph

When $c < -1$ or $1 < c$, then the graph when $z = c$ is a circle





hyperboloid of 2 sheets

Example 7.7. Graph $\frac{x^2}{4}+\frac{y^2}{9}+\frac{z^2}{16}=1$ in three space.

The cross section when $x=0$ is the ellipse $\frac{y^2}{9}+\frac{z^2}{16}=1$.

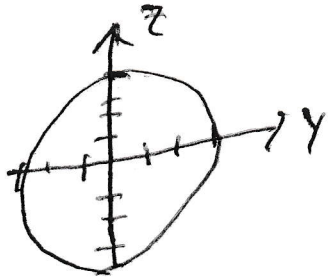
The cross section when $y=0$ is the ellipse $\frac{x^2}{4}+\frac{z^2}{16}=1$.

The cross section when $z=0$ is the ellipse $\frac{x^2}{4}+\frac{y^2}{9}=1$.

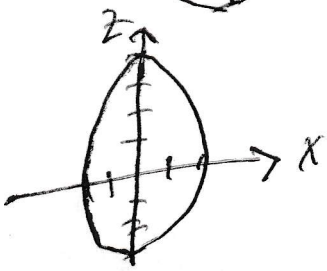
The graph of the surface in 3-space is called an ellipsoid.

Graph $\frac{x^2}{4} + \frac{y^2}{9} + \frac{z^2}{16} = 1$ in 3-space

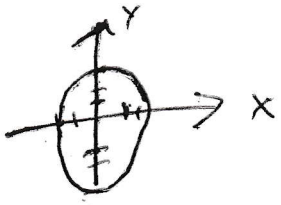
when $x = 0$ the graph $\frac{y^2}{9} + \frac{z^2}{16} = 1$ is an ellipse

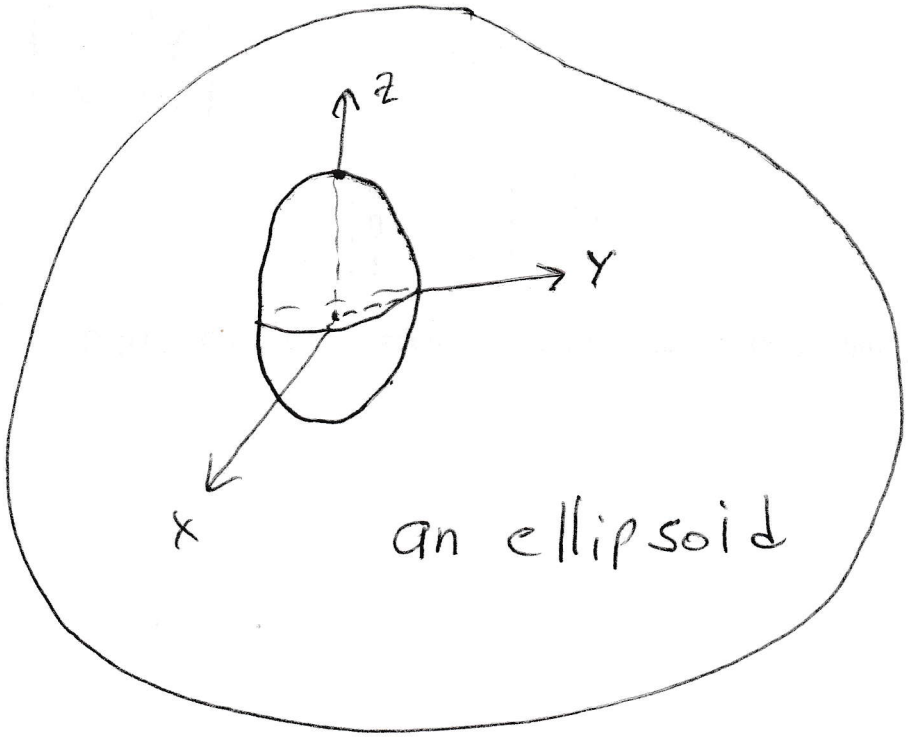


when $y = 0$ $\frac{x^2}{4} + \frac{z^2}{16} = 1$ is an ellipse



when $z = 0$ $\frac{x^2}{4} + \frac{y^2}{9} = 1$ is an ellipse





8. **Section 13.1. Curves in three space.**

One usually describes a curve parametrically:

$$\begin{cases} x = x(t) \\ y = y(t) \\ z = z(t) \end{cases}$$

Example 8.1. Describe the circle $x^2 + y^2 = 1$ parametrically.

There are all kinds of clumsy things one can do, but the easiest way to describe a point on a circle is to describe the coordinates of a point in terms of angle. (Please look at the picture on the next page.) Trigonometry was cooked up to parameterize circles.

$$\boxed{\begin{cases} x = \cos t \\ y = \sin t, \end{cases} \quad \text{with } -\infty < t < \infty.}$$

Example 8.2. Describe the line segment from $P_0 = (1,1)$ to $P_1 = (3,4)$ parametrically. The vector $\overrightarrow{P_0P_1} = 2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}$. Here is our plan. At $t = 0$ we stand on P_0; at $t = 1$ we stand on P_1. At some random time t we stand on the point $P_t = (x,y)$ so that $\overrightarrow{P_0P_t} = t\overrightarrow{P_0P_1}$. So,

$$\overrightarrow{(1,1)(x,y)} = t\overrightarrow{(1,1)(3,4)}.$$

That is,

$$(x-1)\vec{\boldsymbol{i}} + (y-1)\vec{\boldsymbol{j}} = t(2\vec{\boldsymbol{i}} + 3\vec{\boldsymbol{j}}).$$

Our line segment is

$$\boxed{\begin{cases} x = 1 + 2t \\ y = 1 + 3t, \end{cases} \quad \text{with } 0 \le t \le 1.}$$

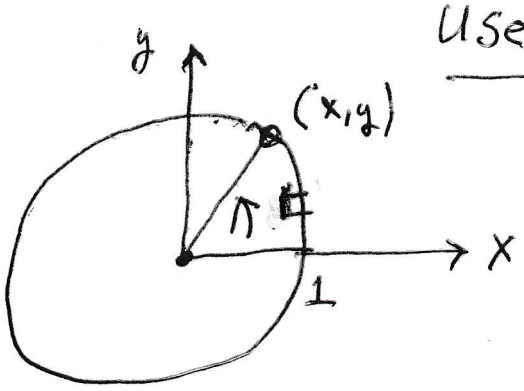
There is a pretty picture on the next page.

Example 8.3. Graph the curve parameterized by

$$\begin{cases} x = \cos t \\ y = \sin t \\ z = t. \end{cases}$$

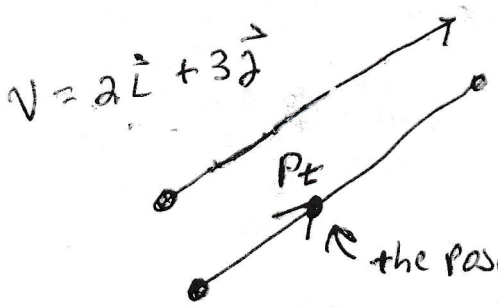
If one forgets about $z = t$, then the curve is the circle $x^2 + y^2 = 1$ as we saw in Example 8.1. So we want to move around and around the **cylinder** $x^2 + y^2 = 1$ in three-space and we want our z coordinate to increase as time increases. The resulting curve is a helix. (It looks like a spiral staircase.)

Use angle Parameterize Circles



$$\begin{cases} X = \cos t \\ Y = \sin t \end{cases}$$

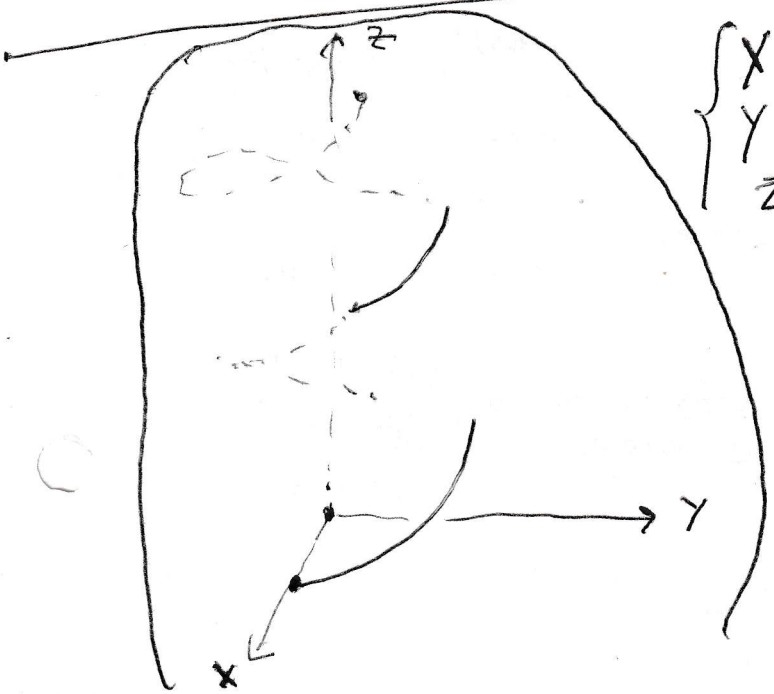
Example 8.1



$P_0 = (1,1)$

$$\begin{cases} X = 1 + 2t \\ Y = 1 + 3t \end{cases} \quad \text{for } 0 \le t \le 1$$

Example 8.2



$$\begin{cases} X = \cos t \\ Y = \sin t \\ Z = t \end{cases}$$

is a helix.

Example 8.3

Example 8.4. Describe the curve $y = f(x)$ parametrically. This is really easy.

$$\begin{cases} x = x \\ y = f(x). \end{cases}$$

The point is if you are willing to parameterize, then you can do calculus on any curve you studied in first semester calculus, as well as circles, and helixes, etc.

If one would rather have one equation rather than a system of three equations, then one can use vector notation:

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k},$$

where $\vec{r}(t)$ is viewed as a position vector. The tail of $\vec{r}(t)$ is **NAILED TO THE ORIGIN.** (Most vectors can be moved. But the tail of a position vector is nailed to the origin.)

Example 8.1 becomes $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$;
Example 8.2 becomes $\vec{r}(t) = (1+2t)\,\vec{i} + (1+3t)\,\vec{j}$;
Example 8.3 becomes $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j} + t\,\vec{k}$; and
Example 8.4 becomes $\vec{r}(t) = t\,\vec{i} + f(t)\,\vec{j}$,
The main advantage of the position vector notation is that if

$$\vec{r}(t) = x(t)\,\vec{i} + y(t)\,\vec{j} + z(t)\,\vec{k}$$

is the position vector of an object at time t, then

$$\vec{r}\,'(t) = x'(t)\,\vec{i} + y'(t)\,\vec{j} + z'(t)\,\vec{k}$$

is the velocity vector of an object at time t (The velocity vector is tangent to the path of the object at time t and the length of the velocity vector at time t, $|\vec{r}\,'(t)|$, is the speed of the object.) and

$$\vec{r}\,''(t) = x''(t)\,\vec{i} + y''(t)\,\vec{j} + z''(t)\,\vec{k}$$

is the acceleration vector of an object at time t.

Example 8.5. Let $\vec{r}(t) = \cos t\,\vec{i} + \sin t\,\vec{j}$.

(a) Find the angle between $\vec{r}(t)$ and $\vec{r}\,'(t)$.
(b) Find the speed of the object.
(c) Find $\vec{r}\,''(t)$.

(a) We calculate

$$\vec{r}\,'(t) = -\sin t\,\vec{i} + \cos t\,\vec{j}.$$

It follows that

$$\vec{r}(t) \cdot \vec{r}\,'(t) = -\cos t \sin t + \sin t \cos t = 0.$$

Thus, $\boxed{\vec{r}(t) \text{ and } \vec{r}\,'(t) \text{ are perpendicular.}}$

(b) The speed of the object is the length of the velocity vector. This is

$$|\vec{\boldsymbol{r}}'(t)| = |-\sin t\,\vec{\boldsymbol{i}} + \cos t\,\vec{\boldsymbol{j}}\,| = \sqrt{(-\sin t)^2 + (\cos t)^2} = \sqrt{\sin^2 t + \cos^2 t} = \sqrt{1} = 1.$$

The object is traveling with constant speed 1.

(c) The acceleration vector is

$$\boxed{\vec{\boldsymbol{r}}''(t) = -\cos t\,\vec{\boldsymbol{i}} - \sin t\,\vec{\boldsymbol{j}}\,.}$$

Our object is running around a circle. The acceleration of our object is always pointed toward the center of the circle.

9. Section 13.2. The equations of motion of an object.

Question 9.1. *Suppose an object starts at the origin and is fired at angle α with initial speed v_0. The acceleration of the object is always $-g\vec{j}$, for some positive constant g. Give the position vector $\vec{r}(t)$ of the object at time t.*

The initial velocity of the object is $\vec{r}\,'(0) = v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}$. (See the picture on the next page, if necessary.)

We are given:

$$\begin{aligned}\vec{r}\,''(t) &= -g\,\vec{j}\\ \vec{r}\,'(0) &= v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}\\ \vec{r}\,(0) &= 0\,\vec{i} + 0\,\vec{j}\end{aligned}$$

We integrate to learn

$$\vec{r}\,'(t) = -gt\,\vec{j} + \vec{c_1}$$

for some constant vector $\vec{c_1}$. Plug in $t = 0$ to learn

$$v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j} = \vec{r}\,'(0) = \vec{c_1}.$$

Thus,

$$\vec{r}\,'(t) = -gt\,\vec{j} + v_0\cos\alpha\,\vec{i} + v_0\sin\alpha\,\vec{j}$$

and

$$\vec{r}\,'(t) = v_0\cos\alpha\,\vec{i} + (v_0\sin\alpha - gt)\,\vec{j}.$$

Integrate again to learn

$$\vec{r}\,(t) = v_0(\cos\alpha)t\,\vec{i} + (v_0(\sin\alpha)t - \tfrac{gt^2}{2})\,\vec{j} + \vec{c_2},$$

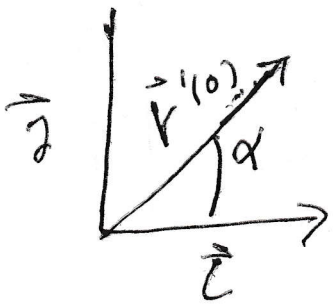
for some constant vector $\vec{c_2}$. Plug in $t = 0$ to learn

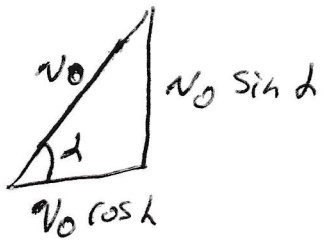
$$0 = \vec{r}\,(0) = \vec{c_2}.$$

Thus,

$$\boxed{\vec{r}\,(t) = v_0(\cos\alpha)t\,\vec{i} + \left(v_0(\sin\alpha)t - \tfrac{gt^2}{2}\right)\vec{j}\,.}$$

The picture for 9.1





we drew $\vec{r}'(0)$ which has length v_0.

so $\vec{r}'(0) = v_0 \cos\alpha\, \vec{i} + v_0 \sin\alpha\, \vec{j}$

10. **Section 13.3. Arc length.**

Fact 10.1. *The length of the curve parameterized by the position vector* $\vec{\boldsymbol{r}}(t)$ *is*

$$\int_{\text{the beginning}}^{\text{the end}} |\vec{\boldsymbol{r}}'(t)|dt.$$

This is easy to understand. One integrates speed to get the distance traveled. In the most elementary situation, (when the path of the object is a straight line and the speed is constant) distance is equal to rate times time. In more complicated situations (where the speed is not necessarily constant and the path is not necessarily a straight line) one uses the more sophisticated version that
the distance traveled is the integral from the beginning to the end of the speed.

Example 10.2. Find the circumference of the circle with center $(0,0)$ and radius 1. (Of course, we know the answer. The circumference of a circle is 2π times the radius. The circle in the present problem has radius 1; so the circumference is 2π.) Lets use the technique of Fact 10.1 to find the answer. The circle is parameterized by

$$\begin{cases} x = \cos t \\ y = \sin t, \end{cases} \qquad \text{with } 0 \le t \le 2\pi.$$

So the circumference of the circle is

$$\begin{aligned} \int_0^{2\pi} |(\cos t\,\vec{\boldsymbol{i}} + \sin t\,\vec{\boldsymbol{j}})'|dt &= \int_0^{2\pi} |-\sin t\,\vec{\boldsymbol{i}} + \cos t\,\vec{\boldsymbol{j}}\,|dt \\ &= \int_0^{2\pi} \sqrt{\sin^2 t + \cos^2 t}\,dt = \int_0^{2\pi} 1\,dt = t\Big|_0^{2\pi} = \boxed{2\pi}. \end{aligned}$$

We did obtain the expected answer.

11. Section 14.1. Functions of several variables.

Example 11.1. $f(x,y) = x^2 + y^2$ is a function of two variables.

(a) What is the domain of f?
(b) What is the image of f?
(c) Sketch the graph of $z = f(x,y)$.
(d) Draw a few level sets of f. That is draw the graph of $f(x,y) = c$ for a few values of c.

(a) A function is a machine. One hands the function some input and it spits out the out put. The question "What is the domain of f?" means what KIND of input can you put into this machine. This particular function will accept any point in the xy-plane. So

The domain of f is the set of points in the xy-plane.

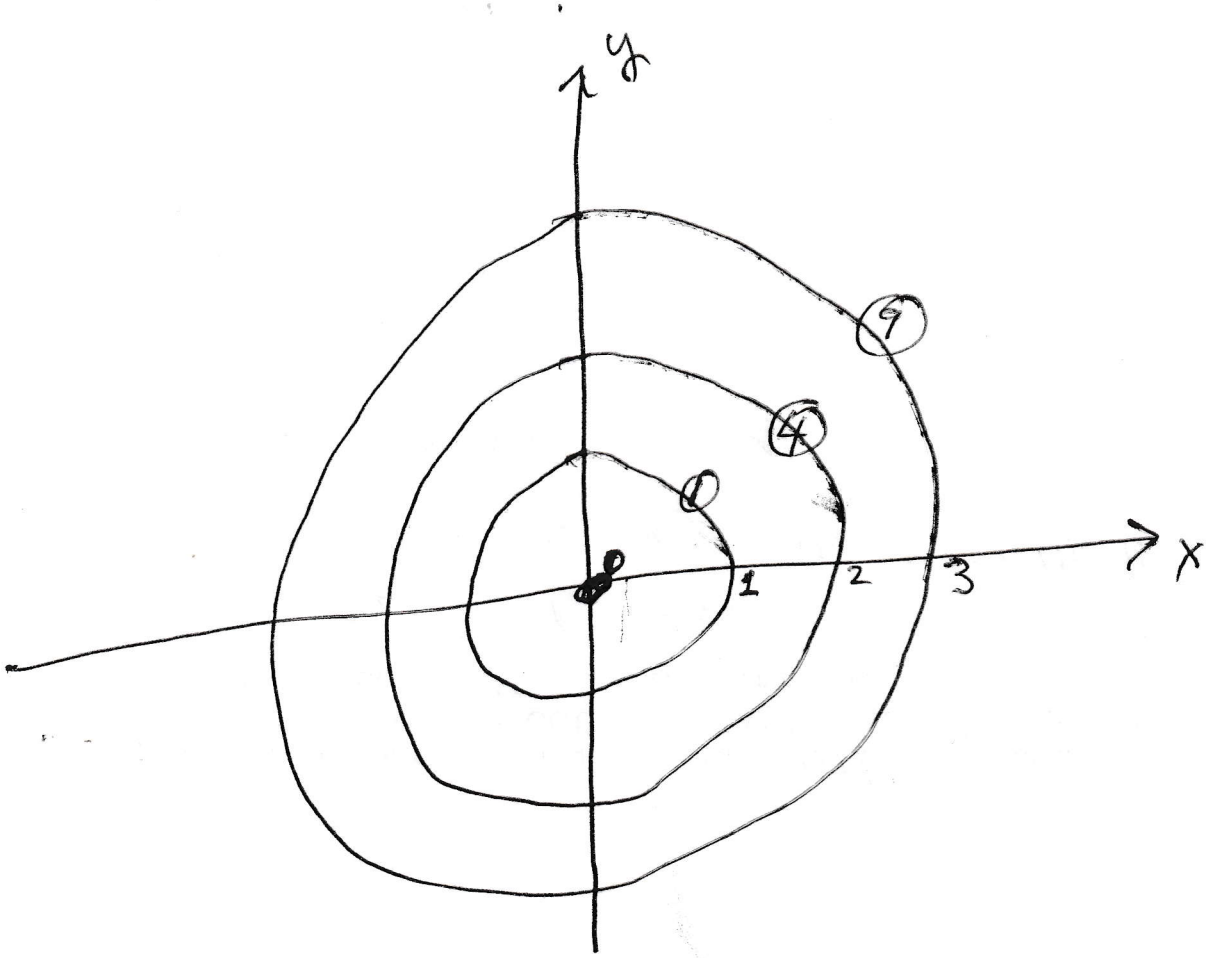
(b) The question "What is the image of f?" means output does f give. (Some people write "range" where I write "image".) At any rate,

the image of f is the set of real numbers which are zero or higher.

(c) Sketch the graph of $z = f(x,y)$ means "graph $z = x^2 + y^2$. This is a paraboloid; see Example 7.3 on page 45.

(d) Often times it is too hard to graph the function $z = f(x,y)$. Instead one draws a level sets of the function. (This is like making a topographic map. One can learn where the hills and valleys are; where the terrain is steep and where the terrain is gentle by looking at a topographic map. One need not look at a three-dimension model of the region where one plans to hike. The level sets of the function in this section are a bunch of concentric circles. One glance at the level sets and one knows that the graph of the function is a giant bowl with the bottom sitting at the origin. There are a few level sets on the next page.

Some level sets for $f(x,y) = x^2+y^2$



The level set $f(x,y) = 4$ is the circle $x^2+y^2 = 4$. This circle has center $(0,0)$ and radius 2.

12. Section 14.2. Limits and Continuity.

LIMITS

Rough Definition: If f is a function and (x_0, y_0) is a point, then

$$\lim_{(x,y)\to(x_0,y_0)} f(x,y) = L$$

means that whenever (x,y) is near (but not equal to) (x_0,y_0), then $f(x,y)$ is near L.

- The key point is that this is a very **bold** statement (because it has to work for all posible meanings for the word "near".) It is most important when $f(x_0,y_0)$ does not make any sense. An important example of a critical limit is $\lim_{x\to 0} \frac{\sin x}{x} = 1$. Another important example is $\lim_{n\to\infty} (1+\frac{1}{n})^n = e$. "Radians" are defined to make the first limit hold. In calculus one measures angles in radians to make the formulas as easy as posible. Similary, the favorite base for exponentiating is base "e", again to make the formulas as easy as posible.
- I did not give you enough information to calculate limits for real.
- The usual way to show that $\lim_{(x,y)\to(x_0,y_0)} f(x,y)$ does not exist is to calculate
  $\lim_{\substack{(x,y)\to(0,0)\\ \text{along some path}}} f(x,y)$ is one number but
  $\lim_{\substack{(x,y)\to(0,0)\\ \text{along some other path}}} f(x,y)$ is a different number.

So, how does one show that a limit does not exist? In first semester calculus one often shows that a limit does not exist by showing that the limit from the left is different than the limit from the right. For example, let $f(x) = \frac{x}{|x|}$. Find $\lim_{x\to 0} f(x)$. The function $f(x)$ is equal to 1 when x is positive and $f(x)$ is equal to -1 when x is negative. Thus,

$$\lim_{x\to 0^+} f(x) = 1 \quad \text{and} \quad \lim_{x\to 0^-} f(x) = -1.$$

The two one-sided limits are different; thus, one concludes that $\lim_{x\to 0} f(x)$ does not exist.

When one finds $\lim_{P\to P_0} f(P)$ when f is a function of several variables, then there are many ways to approach the point P_0. Consider the following problem,

(a) Find $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=3x}} \frac{x^3y}{x^6+2y^2}$.

(b) Find $\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^3}} \frac{x^3y}{x^6+2y^2}$.

(c) Find $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+2y^2}$.

(a) We see that

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=3x}} \frac{x^3y}{x^6+2y^2} = \lim_{x\to 0} \frac{x^3(3x)}{x^6+2(3x)^2} = \lim_{x\to 0} \frac{3x^4}{x^6+18x^2} = \lim_{x\to 0} \frac{3x^4}{x^2(x^4+18)}$$

$$= \lim_{x\to 0} \frac{3x^2}{(x^4+18)} = \frac{0}{18} = \boxed{0.}$$

(b) We see that

$$\lim_{\substack{(x,y)\to(0,0)\\ \text{along } y=x^3}} \frac{x^3y}{x^6+2y^2} = \lim_{x\to 0} \frac{x^3x^3}{x^6+2(x^3)^2} = \lim_{x\to 0} \frac{x^6}{x^6+2x^6} = \lim_{x\to 0} \frac{x^6}{3x^6} = \lim_{x\to 0} \frac{1}{3} = \boxed{\frac{1}{3}}.$$

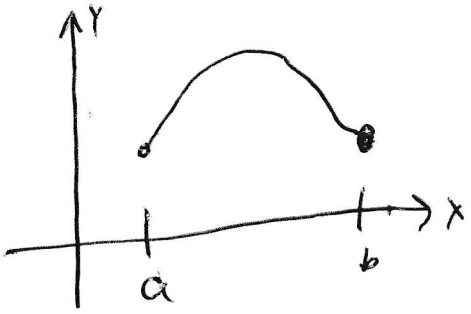
(c) The answers to (a) and (b) are different so $\lim_{(x,y)\to(0,0)} \frac{x^3y}{x^6+2y^2}$ does not exist.

CONTINUITY

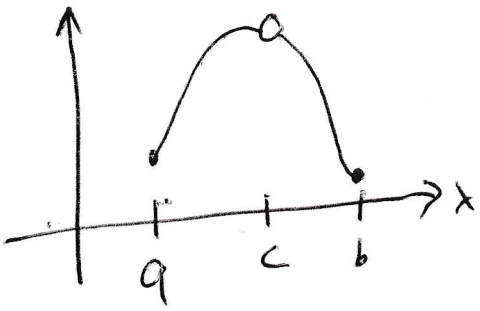
In first semester calculus, one says that a function is continuous on its domain if one can draw its graph without lifting one's pencil. For example, which of the following functions is continuous one $[a,b]$?

Continuity of functions of one variable

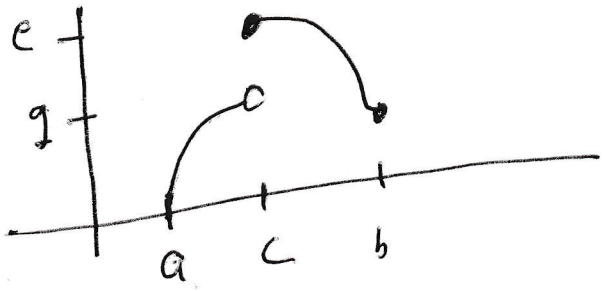
Each picture is the graph of $y = f(x)$.



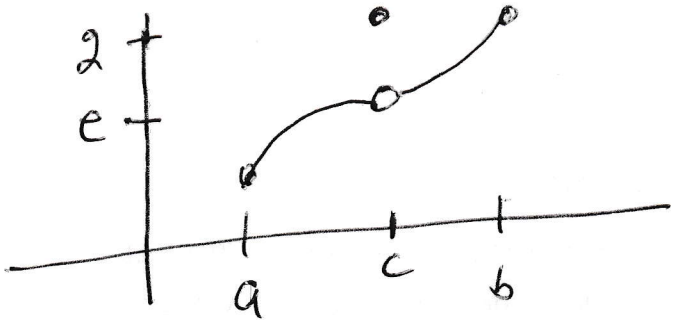
The function is continuous on $[a,b]$



The function is discontinuous at c because $f(c)$ does not exist.



The function is discontinuous at c because $\lim_{x \to c} f(x)$ does not exist. (Notice the $f(c)$ exists and equals e.)



The function is discontinuous at c because

$\lim_{x \to c} f(x) \neq f(c)$

Indeed $f(c) = g$ and

$\lim_{x \to c} f(x) = e$

but $e \neq g$

The main problem with the first approximation of continuity is that it gives no insight into what it means to say that a function of two variables is continuous. Here is the official definition of continuity that works for any number of variables.

Definition The function f is continuous at P_0 if

(a) $f(P_0)$ exists,
(b) $\lim_{P \to P_0} f(P)$ exists, and
(c) $f(P_0) = \lim_{P \to P_0} f(P)$.

Please look at the pictures on the previous page. These three conditions also describe continuity in the one variable case.

I propose the following ideas are things you can take away from the discussion of continuity.

- Most functions one meets in calculus or life **are continuous where ever they are defined.**
- If you are asked "Where is this function continuous?" you should first figure out where the function is defined.
- Usually, if you are dealing with a discontinuous function in life, there are red flags indicating the discontinuity. For example, "... and then a switch was flipped and the power was turned off" is a clear indication of a discontinuous process.

13. Section 14.3. Partial Derivatives

Given $z = f(x,y)$,

(a) to find $\frac{\partial f}{\partial x}$ pretend y is constant and take the derivative with respect to x ($\frac{\partial f}{\partial x}$ is called the partial derivative of f with respect to x), and

(b) to find $\frac{\partial f}{\partial y}$ pretend x is constant and take the derivative with respect to y ($\frac{\partial f}{\partial y}$ is called the partial derivative of f with respect to y) .

Example. Let $f(x,y) = 6x^3y^4 + x\sin y + y\tan x$. Then

$$\frac{\partial f}{\partial x} = 18x^2y^4 + \sin y + y\sec^2 x$$
$$\frac{\partial f}{\partial y} = 24x^3y^3 + x\cos y + \tan x.$$

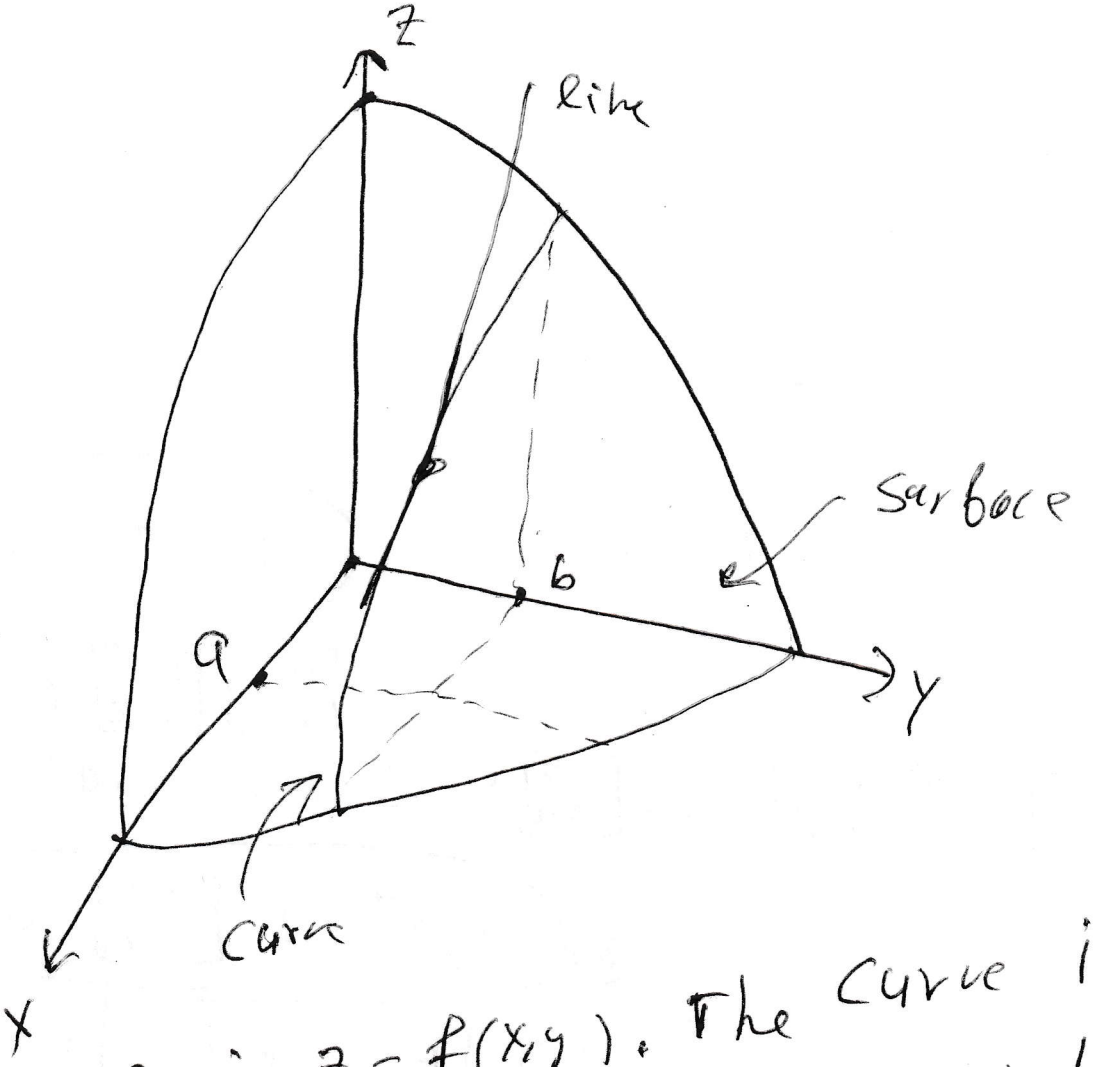
What does it mean?

$\frac{\partial f}{\partial x}|_{(a,b)}$ is the slope of the line tangent to the curve which is the intersection of the surface $z = f(x,y)$ and the plane $y = b$ at the point $(a,b,f(a,b))$.

$\frac{\partial f}{\partial y}|_{(a,b)}$ is the slope of the line tangent to the curve which is the intersection of the surface $z = f(x,y)$ and the plane $x = a$ at the point $(a,b,f(a,b))$.

Please look at the picture on the next page.

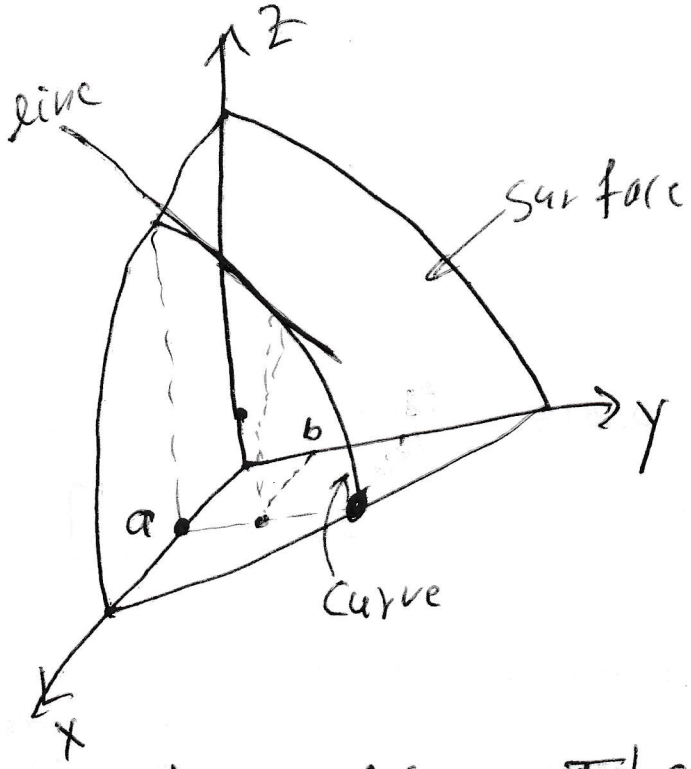
The meaning of Partial Derivative



The surface is $z = f(x,y)$. The curve is the intersection of $z = f(x,y)$ and $y = b$. The line is the line which is tangent to the curve at $(a, b, f(a,b))$.

$\left.\frac{\partial f}{\partial x}\right|_{(a,b)}$ is the slope of the line

The meaning of Partial Derivative



The surface is $z = f(x,y)$. The curve on the surface is the intersection of $z = f(x,y)$ and $x = a$. The line is the line tangent to the intersection of $z = f(x,y)$ and $x = a$ at $(a, b, f(a,b))$.

$\left.\frac{\partial f}{\partial y}\right|_{(a,b)}$ is the slope of the line.

14. Section 14.4. The Chain Rule for functions of several variables.

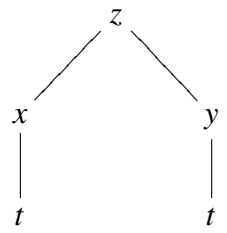
The fact that you are in Math 241 guarantees that you can do the Chain Rule for a function of a single variable. If y is a function of u and u is a function of[4] x, then

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx}.$$

For example, if $y = \sin(x^2)$, you could think $y = \sin u$ and $u = x^2$; so,[5]

$$\frac{dy}{dx} = \frac{dy}{du}\frac{du}{dx} = (\cos u)(2x) = 2x\cos(x^2).$$

What does the Chain Rule look like for functions of several variables? If $z = z(x,y)$, $x = x(t)$, and $y = y(t)$, then there are two contributions to $\frac{dz}{dt}$; namely $\frac{\partial z}{\partial x}\frac{dx}{dt}$ and $\frac{\partial z}{\partial y}\frac{dy}{dt}$. I find the picture



to be helpful. In order to calculate $\frac{dz}{dt}$ correctly, one must ADD the two contributions[6]:

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

The best way to convince you of this is to work at an example two ways; once using the chain rule and once not using the chain rule and then observing that the same answer was obtained each time.

Example. Suppose $w = x^2 + y^2$, $x = \cos t + \sin t$, and $y = \cos t - \sin t$. Compute $\frac{dw}{dt}$ two ways.

(a) In the first way, express w in terms of t without any x's or y's. Then directly calculate $\frac{dw}{dt}$.

(b) In the second way, first use the Chain Rule for functions of several variables, then write every remaining x and y in terms of t.

---

[4] The usual shorthand for these words is "If $y = y(u)$ and $u = u(x)$".

[5] Please notice that u was not part of the original problem; so u is not allowed to be part of the answer.

[6] I should comment about the fact that sometimes I have written d and sometimes I have written ∂. The symbol z is a function of the two variables x and y. When I take a derivative of z with respect to either x or y, that derivative is in fact a partial derivative. However at the level of t, z depends only on t and so do x and y; consequently the derivative of z, or x, or y with respect to t all are ordinary derivatives. This is not a huge point. If one messes it up, almost everyone will figure out what was meant. But using the wrong symbol is like making a spelling mistake or a grammar mistake. Almost everyone knows what was meant, but a sense of indifference or ignorance was communicated along with the content.

(a) Observe that

$$w = (\cos t + \sin t)^2 + (\cos t - \sin t)^2.$$

Thus,

$$w = \cos^2 t + 2\cos t \sin t + \sin^2 t + \cos^2 t - 2\cos t \sin t + \sin^2 t.$$

The terms $+2\cos t \sin t - 2\cos t \sin t$ add to zero. The terms $\cos^2 + \sin^2 t$ both add to 1, and

$$w = 2.$$

We compute that

$$\boxed{\frac{dw}{dt} = 0.}$$

(b) Observe that

$$\frac{dw}{dt} = \frac{\partial w}{\partial x}\frac{dx}{dt} + \frac{\partial w}{\partial y}\frac{dy}{dt}$$

Thus,

$$\frac{dw}{dt} = 2x(-\sin t + \cos t) + 2y(-\sin t - \cos t)$$

$$\frac{dw}{dt} = 2(\cos t + \sin t)(-\sin t + \cos t) + 2(\cos t - \sin t)(-\sin t - \cos t)$$

$$\frac{dw}{dt} = 2(\cos^2 t - \sin^t) + 2(\sin^2 t - \cos^2 t) = \boxed{0.}$$

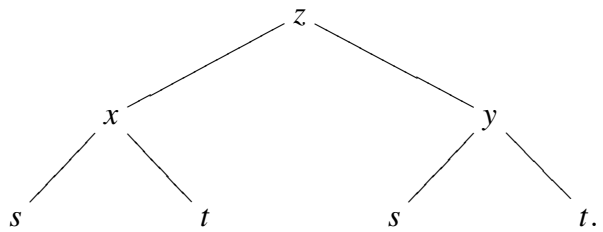
Here is one more comment, which is probably obvious. If $z = z(x,y)$, $x = x(s,t)$, and $y = y(s,t)$, then

$$\frac{\partial z}{\partial s} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial s} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial s}$$

and

$$\frac{\partial z}{\partial t} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial t} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial t}.$$

In this case, the helpful picture looks like



15. Sections 14.5 and 14.6. Gradients, directional derivatives, and tangent planes.

Suppose $f = f(x,y)$. We saw that $\frac{\partial f}{\partial x}|_{(a,b)}$ measures the rate of change of f if I start at the point (a,b) and walk in the direction $\overrightarrow{i}$ and $\frac{\partial f}{\partial y}|_{(a,b)}$ measures the rate of change of f if I start at the point (a,b) and walk in the direction $\overrightarrow{j}$.

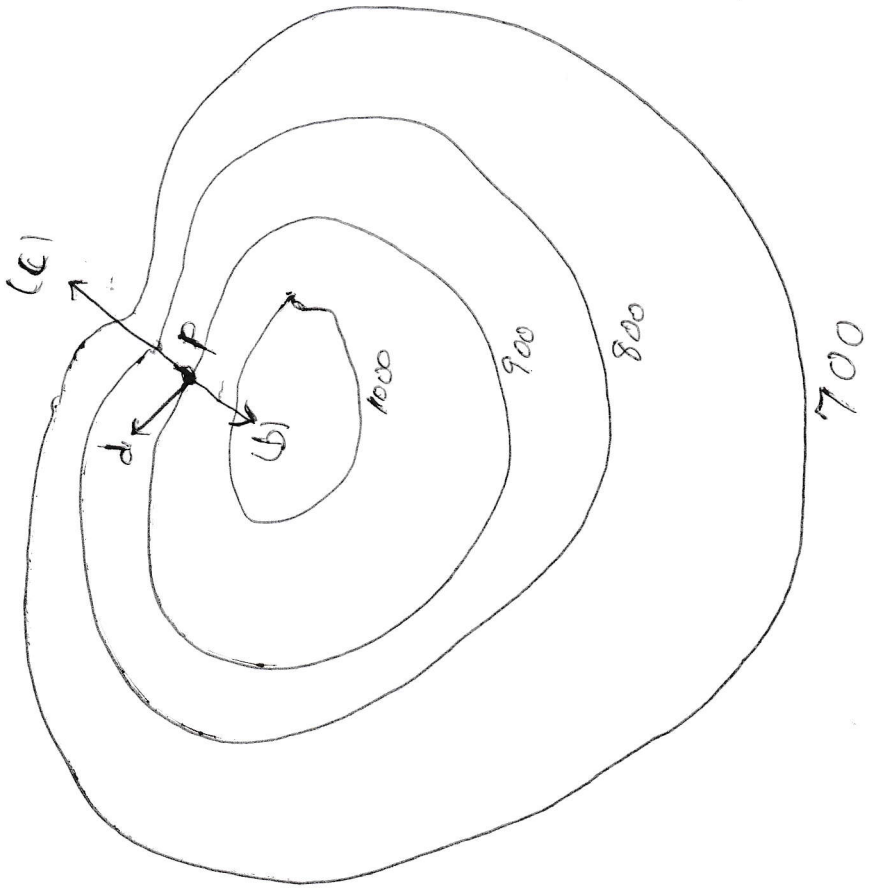
It is natural to ask the following questions?

(a) How do I find the rate of change of f if I start at the point (a,b) and walk in some other direction?
(b) I start at the point (a,b). In which direction should I walk if I want f to increase as rapidly as possible?
(c) I start at the point (a,b). In which direction should I walk if I want f to decrease as rapidly as possible?
(d) I start at the point (a,b). I want to walk, but I want f to remain as unchanged as possible. Which way should I go?

The fact of the matter is that if you are looking at a topographic map (i.e. a picture of many level sets of f), then you can answer these questions instantly. The point of sections 14.5 and 14.6 is to answer these questions even if we don't have a topographic map handy.

Here is a collection of level sets. Start at the point P. Notice that the vector labeled (b) gives the steepest climb starting at P, the vector labeled (c) gives the steepest decent starting at P, and the vector labeled (d) gives a direction with virtually no change in f. (The vector labeled (d) is supposed to be tangent to the level set containing P. It also is supposed to be perpendicular to (b) and (c).)

Level sets for $f = f(x,y)$



At an rate, if f is a function, P is a point, and $\vec{v}$ is vector, then

$$D_{\vec{v}} f|_P$$

is the directional derivative of f in the direction of $\vec{v}$ starting at the point P. This symbol, when fully decorated, represents a number. When $\vec{v} = \vec{i}$, then

$$D_{\vec{i}} f|_P = \frac{\partial f}{\partial x}|_P.$$

(Keep in mind that $\frac{\partial f}{\partial x}$ is a function and when this function is evaluated at the point P, one obtains a number.) In a similar manner, when $\vec{v} = \vec{j}$, then

$$D_{\vec{j}} f|_P = \frac{\partial f}{\partial y}|_P.$$

(Keep in mind again that $\frac{\partial f}{\partial y}$ is a function and when this function is evaluated at the point P, one obtains a number.)

In general,

$$D_{\vec{v}} f|_P = \vec{\nabla} f|_P \cdot \frac{\vec{v}}{|\vec{v}|} \tag{15.0.1}$$

where $\vec{\nabla} f$ is called the gradient of f and

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} + \frac{\partial f}{\partial z}\vec{k} \quad \text{if } f = f(x,y,z)$$

and

$$\vec{\nabla} f = \frac{\partial f}{\partial x}\vec{i} + \frac{\partial f}{\partial y}\vec{j} \quad \text{if } f = f(x,y).$$

Notice that $\vec{\nabla} f$ is a "vector-valued function of several variables." In other words, $\vec{\nabla} f$ is a function, the domain of $\vec{\nabla} f$ is a set of points in two space or three space[7] and the answers that $\vec{\nabla} f$ gives are vectors.[8]

I will calculate some $D_{\vec{v}} f|_P$, some $\vec{\nabla} f$, and some $\vec{\nabla} f|_P$ in just a minute. First I must comment on (15.0.1).

The directional derivative $D_{\vec{v}} f|_P$ does not have any thing to do with the length of $\vec{v}$ all that matters is the direction of $\vec{v}$. That is why $\frac{\vec{v}}{|\vec{v}|}$ appears on the right side. Some people won't even write the symbol $D_{\vec{v}} f|_P$ unless $\vec{v}$ has length 1. The ultimate formula is easier for such people, but they have to remember to adjust the input to be appropriate before they plug the input into the easier formula. You can do whichever you like, but be sure to do it correctly.

The formula (15.0.1) also tells us the significance of the gradient vector. Recall that

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\text{the angle between } \vec{a} \text{ and } \vec{b}).$$

[7]This is the meaning of "of several variables"

[8]This is why $\vec{\nabla} f$ is called a "vector-valued" function.

So[9]

$$D_{\vec{\boldsymbol{v}}} f|_P = |\overrightarrow{\nabla} f|_P| \, |\frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}| \cos(\text{the angle between } \overrightarrow{\nabla} f|_P \text{ and } \vec{\boldsymbol{v}}).$$

The number $|\frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}|$ is always 1; the function cosine roams between -1 and plus 1.

We conclude that

(a) The gradient vector $\overrightarrow{\nabla} f|_P$ points in the direction of the largest directional derivative of f at P.

(b) The length of the gradient vector is the largest directional derivative of f at P.

(c) Minus the gradient vector $\overrightarrow{\nabla} f|_P$ points in the direction of the smallest directional derivative of f at P and minus the length of the gradient vector is the smallest directional derivative of f at P.

(d) **GRADIENTS ARE PERPENDICULAR TO LEVEL SETS.**

The assertion (d) can be thought of as Gradients are fleeing the level set as expediently as possible. The direction which flees the status quo as expediently is the direction perpendicular to the status quo.

Assertion (d) can also be thought of this way. Any direction which is perpendicular to the gradient has no change in f. In other words, if one is walking in the domain of f and one's direction is perpendicular to the gradient of f, then one is leisurely walking along along a level set of f.

Here is an example:

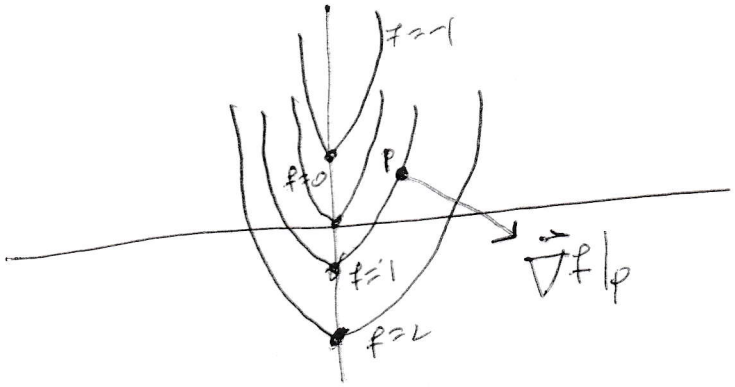
[9]The right side of this display is the length of the gradient vector evaluated at P times the length of the unit vector $\frac{\vec{\boldsymbol{v}}}{|\vec{\boldsymbol{v}}|}$ times the cosine of the angle between the two vectors.

Example Consider $f(x,y) = x^2 - y$ $P = (\sqrt{2}, 1)$

(a) Draw many level sets for $f(x,y) = c$

(b) Draw the level set of f that contains P.
Draw $\vec{\nabla} f|_P$. Put the tail on P

(c) Find the equation of the line tangent to the level set of f at P



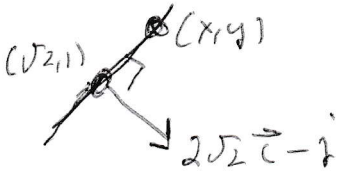
$f(x,y) = c$ is
$x^2 - y = c$
or $x^2 - c = y$

The level set of f that contains $(\sqrt{2}, 1)$
is $(\sqrt{2})^2 - 1 = c$

so $2 - 1 = c$
that is $1 = c$

$\vec{\nabla} f|_P = (2x\vec{i} - \vec{j})|_P = 2\sqrt{2}\,\vec{i} - \vec{j}$

Of course, you can use Math 141 techniques to find the equation of the line tangent to $y = x^2 - 1$ at $(\sqrt{2}, 1)$.
I will use 241 techniques. The point (x,y) is on the tangent line if and only if



(x,y) is on the tangent line if and only if

$$\overrightarrow{(\sqrt{2},1)(x,y)} \cdot (2\sqrt{2}\vec{i} - \vec{j}) = 0$$

$$\left((x-\sqrt{2})\vec{i} + (y-1)\vec{j}\right) \cdot (2\sqrt{2}\vec{i} - \vec{j}) = 0$$

$$2\sqrt{2}(x-\sqrt{2}) - (y-1) = 0$$

$$2\sqrt{2}x - 3 = y$$

(c)

Here is another example.

Example Find the equation of the plane tangent to $z = x^2 + y^2$ at $(1,2,5)$. (Of course the surface $z = x^2 + y^2$ is just a paraboloid. Draw the parabola $z = y^2$ in the yz-plane. Now revolve the parabola about the z-axis. I'll put a picture at the end of the example.)

At any rate, gradients are perpendicular to level sets. So we view the given equation as a level set. That is, we move z to the other side. Now our surface is level zero of the function $f(x,y,z) = x^2 + y^2 - z$. Our tangent plane is perpendicular to the vector

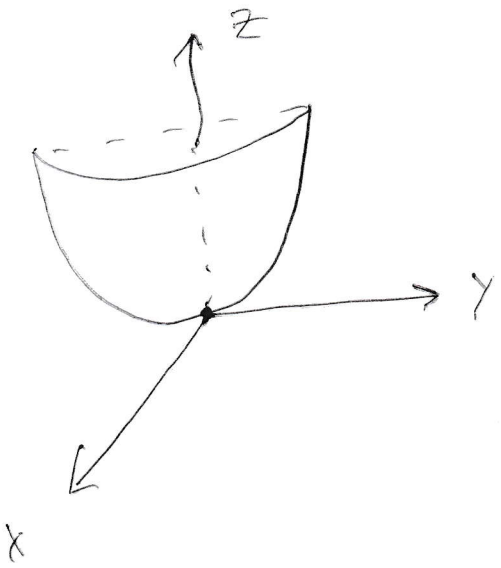
$$(\vec{\nabla} f)|_{(1,2,5)} = (\frac{\partial f}{\partial x}\vec{\boldsymbol{i}} + \frac{\partial f}{\partial y}\vec{\boldsymbol{j}} + \frac{\partial f}{\partial z}\vec{\boldsymbol{k}})|_{(1,2,5)} = (2x\vec{\boldsymbol{i}} + 2y\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}})|_{(1,2,5)} = 2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}.$$

The equation of the plane through $(1,2,5)$ and perpendicular to $2\vec{\boldsymbol{i}} + 4\vec{\boldsymbol{j}} - \vec{\boldsymbol{k}}$ is

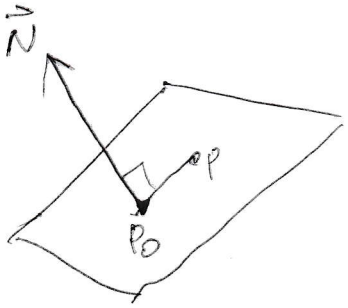
$$\boxed{2(x-1) + 4(y-2) - (z-5) = 0.}$$

I promised a picture of the paraboloid $x^2 + y^2 = z$. I will include for free how I remember the equation of the plane through (x_0, y_0, z_0) perpendicular to $\vec{\boldsymbol{N}} = A\vec{\boldsymbol{i}} + B\vec{\boldsymbol{j}} + C\vec{\boldsymbol{k}}$.

$z = x^2 + y^2$



The plane through $P_0 = (x_0, y_0, z_0)$ perpendicular to $\vec{N} = A\vec{i} + B\vec{j} + C\vec{k}$ is



If $P = (x, y, z)$ is on the plane, then

$\overrightarrow{P_0P} \cdot \vec{N} = 0$

So $A(x - x_0) + B(y - y_0) + C(z - z_0) = 0$

16. Section 14.7. Extreme points for functions of two variables.

There are two two parts to this section:

A. Local extreme points.
B. Absolute extreme points.

16.A. **Local extreme points.** The problem we study is: given a differentiable function $z = f(x,y)$, find all points (a,b) such that

(a) $f(x,y) \leq f(a,b)$ for all points (x,y) near (a,b) (in this case, $(a,b,f(a,b))$ is a local maximum point of the graph of f), or
(b) $f(a,b) \leq f(x,y)$ for all points (x,y) near (a,b) (in this case, $(a,b,f(a,b))$ is a local minimum point of the graph of f), or
(c) the plane tangent to $z = f(x,y)$ at $(a,b,f(a,b))$ is parallel to the xy-plane, in some direction $(a,b,f(a,b))$ is a local minimum point, and in some other direction $(a,b,f(a,b))$ is a local maximum point (in this case $(a,b,f(a,b))$ is called a saddle point of the graph of $z = f(x,y)$).

The next page contains three pictures.

- The origin is a local minimum point of $z = x^2 + y^2$.
- The origin is a local maximum point of $z = -(x^2 + y^2)$.
- The origin is a saddle point of $z = y^2 - x^2$. The plane tangent to $z = y^2 - x^2$ at the origin is the xy-plane. If one looks at the intersection of $z = y^2 - x^2$ and the plane $x = 0$, then one is looking at $z = y^2$ which clearly has a local minimum at the origin. If one looks at the intersection of $z = y^2 - x^2$ and the plane $y = 0$, then one is looking at $z = -x^2$ which clearly has a local maximum at the origin.

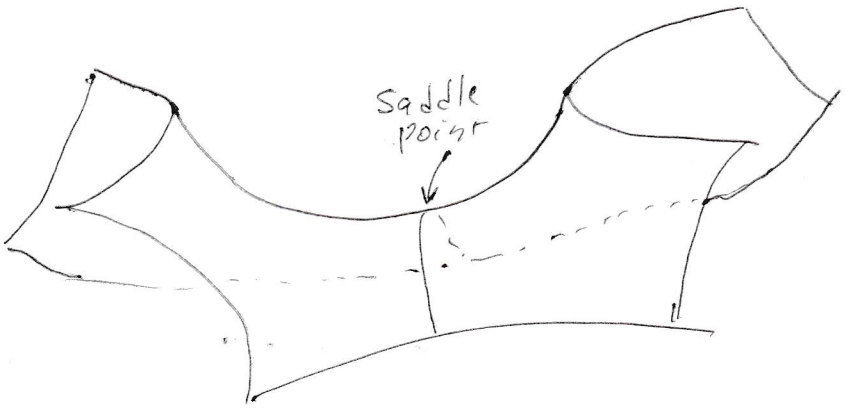
Local Minimum, Local Maximum, Saddle Point

$z = x^2 + y^2$

local minimum

local maximum

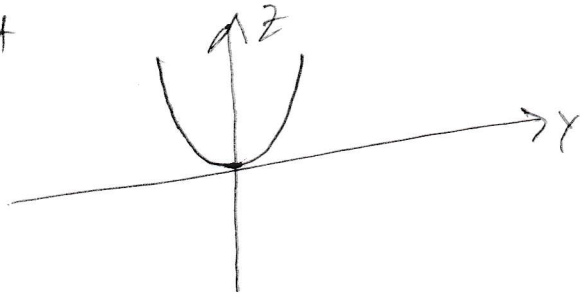
$z = -(x^2 + y^2)$



$z = y^2 - x^2$

If one looks at the intersection of $z = y^2 - x^2$ and $x = 0$

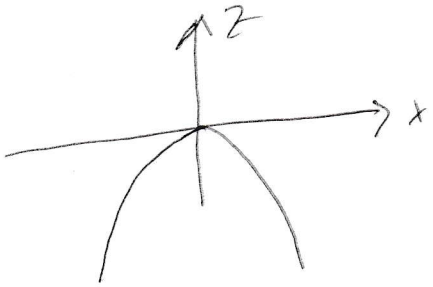
One looks at



which has a minimum at the origin

If one looks at the intersection of $z = y^2 - x^2$ and $y = 0$

One looks at



which has a maximum at the origin.

In first semester calculus one finds the local extreme points of $y = f(x)$ by finding all points where $\frac{dy}{dx} = 0$. These points are called critical points. Then one applies the second derivative test at each critical point.

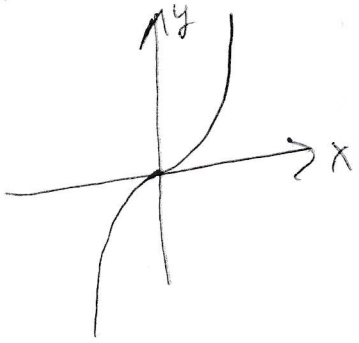
Here is the second derivative test from first semester calculus.

- If x_0 is a critical point of $y = f(x)$ and $0 < f''(x_0)$, then ($y = f(x)$ is concave up at x_0) and $(x_0, f(x_0))$ is a local minimum of $y = f(x)$.
- If x_0 is a critical point of $y = f(x)$ and $f''(x_0) < 0$, then ($y = f(x)$ is concave down at x_0) and $(x_0, f(x_0))$ is a local maximum of $y = f(x)$.
- If x_0 is a critical point of $y = f(x)$ and $f''(x_0) = 0$, then you will have to use some other test to figure out if $(x_0, f(x_0))$ is really a local extreme point of $y = f(x)$ and what kind of local extreme point $(x_0, f(x_0))$ is.

It is probably worth pointing our that if x_0 is a critical point of $y = f(x)$ and $f''(x_0) = 0$, then anything is possible. In particular, $f(x) = x^3$, $f(x) = x^4$, and $f(x) = -x^4$ all satisfy $f'(0) = 0$ and $f''(0) = 0$ but $y = x^3$ does not have a local extreme point at the origin, $y = x^4$ has a local minimum at the origin, and $y = -x^4$ has a local maximum at the origin. I have sketched this pictures on the next page.

Sometimes the second derivative test is not enough

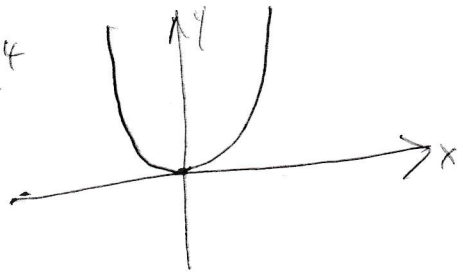
$y = x^3$



$y'(0) = 0 \quad y''(0) = 0$

$(0,0)$ is neither a local maximum nor a local minimum

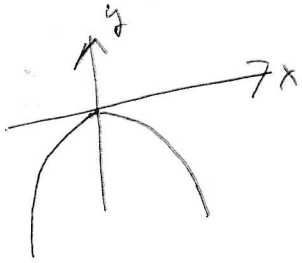
$y = x^4$



$y'(0) = 0 \quad y''(0) = 0$

$(0,0)$ is a local minimum

$y = -x^4$



$y'(0) = 0 \quad y''(0) = 0$

$(0,0)$ is a local maximum

To find the local extreme points for a function of two variables, we do more or less the same thing as was done for a function of one variable. There are two changes.

- We have to deal with two partial derivatives instead of one ordinary derivative.
- The second derivative test is a little more complicated because it has to deal with saddle points which do not occur for functions of one variable.

The procedure for finding the local extreme points and saddle points of $z = f(x,y)$.

Step 1. Find all points (a,b) where both partial derivatives $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are zero. Such points (a,b) are called the critical points of f.

Step 2. Apply the Second Derivative test at each critical point (a,b) of f.

The Second Derivative test for a function of two variables. Suppose (a,b) is a critical point of the function $z = f(x,y)$. Let[10]

$$H = \left(\frac{\partial^2 f}{\partial x^2} \frac{\partial^2 f}{\partial y^2} - \left(\frac{\partial^2 f}{\partial x \partial y} \right)^2 \right) \Bigg|_{(a,b)}.$$

- If $H < 0$, then $(a,b,f(a,b))$ is a saddle point of $z = f(x,y)$.
- If $0 < H$ and $0 < \frac{\partial^2 f}{\partial x^2}|_{(a,b)}$, then $(a,b,f(a,b))$ is a local minimum of $z = f(x,y)$.
- If $0 < H$ and $\frac{\partial^2 f}{\partial x^2}|_{(a,b)} < 0$, then $(a,b,f(a,b))$ is a local maximum of $z = f(x,y)$
- If $H = 0$, then you have to use another test.

I think of the second derivative test this way. The Hessian separates the saddle point from the honest local extreme points. Once one is certain that one is looking at an honest local extreme point, then the second derivative test from first semester calculus kicks in. If you are standing on an honest local extreme point and the graph is concave down, then you are standing on a local minimum. If you are standing on an honest local extreme point and the graph is concave up, then you are standing on a local minimum.

Do notice that H is $f_{xx}(a,b)f_{yy}(a,b)$ minus a perfect square. If H is positive, then $f_{xx}(a,b)$ and $f_{yy}(a,b)$ have the same sign.

At our favorite saddle point (the origin on $z = y^2 - x^2$), the two pure partial derivatives f_{xx} and f_{yy} have opposite signs; hence H is negative. If we rotated the saddle surface a little bit, keeping the z-axis fixed, the equation of the rotated saddle surface would be a little different. It is not immediately obvious that the new f_{xx} and the new f_{yy} have opposite signs; but the new f_{xy}^2 term in the Hessian would play a more important role and the Hessian would still be negative.

[10]The Hessian H is the product of the pure second partial derivatives minus the square of the mixed second partial derivative evaluated at the point (a,b).

If the Hessian is zero, you really do not know what you have. The three functions $f(x,y) = x^4 + y^4$, $f(x,y) = -(x^4 + y^4)$, and $f(x,y) = y^4 - x^4$ all have critical points at the origin and all have H equal zero at the origin; but $f(x,y) = x^4 + y^4$ has a local minimum at the origin, $f(x,y) = -(x^4 + y^4)$ has a local maximum at the origin, and $f(x,y) = y^4 - x^4$ has a saddle point at the origin.[11] On the other hand, you should not spend a lot (i.e., any) time worrying about $H = 0$. We have to leave things for you to learn after you complete this class.

You should do the calculation to find the local extreme points and saddle points of $f(x,y) = x^2 + y^2$, $f(x,y) = -(x^2 + y^2)$, and $f(x,y) = y^2 - x^2$. These are easy and you know what the answers are.

I will type a more exotic problem.

Example Find all local maximum points, local minimum points, and all saddle points of $f(x,y) = x^3 + 3xy^2 - 15x + y^3 - 15y$.

We calculate $f_x = 3x^2 + 3y^2 - 15$ and $f_y = 6xy + 3y^2 - 15$.

We solve the system of equations

$$\begin{cases} 0 = 3x^2 + 3y^2 - 15 \\ 0 = 6xy + 3y^2 - 15 \end{cases}$$

simultaneously. The solution set of our system of equations is unchanged if the second equation is replaced by the first equation minus the second equation. We solve the system of equations

$$\begin{cases} 0 = 3x^2 + 3y^2 - 15 \\ 0 = 3x^2 - 6xy \end{cases}$$

simultaneously. We solve the system of equations

$$\begin{cases} 0 = 3x^2 + 3y^2 - 15 \\ 0 = 3x(x - 2y) \end{cases}$$

simultaneously. If $x = 0$ then $y = \pm\sqrt{5}$. So two critical points are $(0, \sqrt{5})$ and $(0, -\sqrt{5})$. To find the other critical points we solve

$$\begin{cases} 0 = 3x^2 + 3y^2 - 15 \\ 0 = x - 2y \end{cases}$$

simultaneously. We solve

$$\begin{cases} 0 = 3(2y)^2 + 3y^2 - 15 \\ x = 2y \end{cases}$$

[11] The graph of $f(x,y) = x^4 + y^4$ looks somewhat similar to the graph of $f(x,y) = x^2 + y^2$. The analogous statements hold for the other two surfaces.

simultaneously. We solve

$$\begin{cases} 0 = 15y^2 - 15 \\ x = 2y \end{cases}$$

simultaneously. The critical points are $(0,\sqrt{5})$, $(0,-\sqrt{5})$, $(2,1)$, and $(-2,-1)$. We must apply the second derivative test at each critical point.

We calculate $f_{xx} = 6x$, $f_{xy} = 6y$, and $f_{yy} = 6x+6y$. We see that

$$f_{xx}f_{yy} - f_{xy}^2 = 36(x(x+y) - y^2) = 36(x^2 + xy - y^2).$$

Observe that at $(0,\sqrt{5})$ and $(0,-\sqrt{5})$ the Hessian is

$$36(0+0-5),$$

which is negative. We conclude at this point that $(0,\sqrt{5},f(0,\sqrt{5}))$ and $(0,-\sqrt{5},f(0,-\sqrt{5}))$ are both saddle points of $z = f(x,y)$.

At the critical point $(2,1)$, the Hessian is

$$36(4+2-1)$$

and this is positive. Furthermore $f_{xx}(2,1) = 12$, which is also positive. We conclude at this point that $(1,2,f(1,2))$ is a local minimum point of $z = f(x,y)$. (If it is useful to you, my mind processes: I am standing on a local extreme point the second derivative is positive; so the graph is concave up. I must be standing on a local minimum.)

At the critical point $(-2,-1)$, the Hessian is

$$36(4+2-1)$$

and this is positive. Furthermore $f_{xx}(-2,-1) = -12$, which is negative. We conclude at this point that $(-2,-1,f(-2,-1))$ is a local maximum point of $z = f(x,y)$. (If it is useful to you, my mind processes: I am standing on a local extreme point the second derivative is negative; so the graph is concave down. I must be standing on a local maximum.)

At any rate the answer to the problem is:

$(0,\sqrt{5},f(0,\sqrt{5}))$ is a saddle point of $z = f(x,y)$;
$(0,-\sqrt{5},f(0,-\sqrt{5}))$ is a saddle points of $z = f(x,y)$;
$(2,1,f(2,1))$ is a local minimum point of $z = f(x,y)$; and
$(-2,-1,f(-2,-1))$ is a local maximum point of $z = f(x,y)$.

16.B. **Absolute extreme points.** Today we study absolute extreme points from 14.7. Humans are very fond absolute extreme points. We want to find the maximum possible profit; or we want to find the maximum possible strength; or we want to find the minimum possible risk, etc.. Consequently, this is a very important unit.

Lets first recall the first semester calculus version of the problem. The mathematical basis for the calculation is the following Theorem.

Theorem. A continuous function defined on a closed interval has an absolute maximum and an absolute minimum.

The standard max-min problem in first semester calculus is: Let $f(x)$ be a continuous function defined for $a \leq x \leq b$. Find the absolute maximum points and the absolute minimum points of $y = f(x)$ with $a \leq x \leq b$.

The plan of attack is based on the fact that the the absolute extreme points occur either in the interior or on the boundary. If an absolute extreme point occurs in the interior than it must occur at a local extreme point (hence at a critical point where $f'(x) = 0$ or $f'(x)$ does not exist.) We collect all of the critical points; we also collect both boundary points. The maximum occurs at one of these and the minimum also occurs at one of these. We plug each critical point and each boundary point into f and determine largest and smallest value of f. We label the corresponding points as the absolute maximum and absolute minimum points of $y = f(x)$ for $a \leq x \leq b$.

Here is an example. Find the absolute maximum and absolute minimum of $f(x) = 5 - x^2$ with $-2 \leq x \leq 1$. We see that $f'(x) = -2x$. Thus $f'(x) = 0$ when $x = 0$. We plug $x = -2$, $x = 0$, and $x = 1$ into f

$$f(-2) = 5 - (-2)^2 = 1$$
$$f(0) = 5 - (0)^2 = 5$$
$$f(1) = 5 - (1)^2 = 4$$

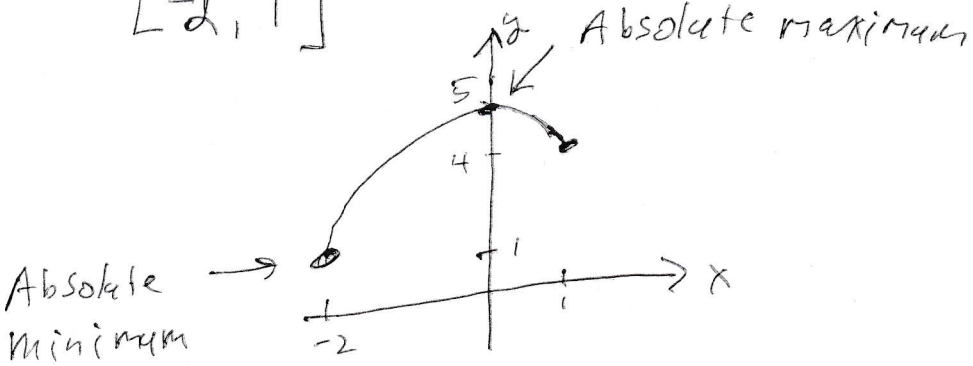
We observe that 5 is the largest number among $1, 5, 4$ and 1 is the smallest. We conclude that

$(0,5)$ is the absolute maximum of $y = f(x)$ on $[-2,1]$ and
$(-2,1)$ is the absolute minimum of $y = f(x)$ on $[-2,1]$.

I will put a picture on the next page.

Absolute Extreme Points

Find the absolute extreme points of $f(x) = 5 - x^2$ on $[-2, 1]$



Absolute extreme points of functions of two variables.

We start with the mathematical basis for the project.

Theorem. A continuous function of two variables which is defined on a closed bounded region in the xy-plane has an absolute maximum and an absolute minimum.

A region in the plane is bounded if one can draw a giant circle around it. (The point is that the region does not go off to infinity; it lives inside some giant circle.) A region in the plane is closed if it contains its own boundary. We dealt with the word continuous in section 14.2.

Please do not worry about any of the above words. I will tell you the region of interest by telling you its boundary. The instructions about the region will look like "the region which is bounded by the circle ...", or "the region which is bounded by the triangle ...", or "the region which is bounded by the curves ... ".

The goal of today's lecture is: We are given a continuous function $f(x,y)$ which is defined on a closed bounded region R in the xy-plane. We want to find the absolute maximum and the absolute minimum of f on R.

The approach in 241 is similar to the approach of first semester calculus.

Step 1. Find all points in the interior of the region where both partial derivatives of f vanish (or at least one of the partial derivatives does not exist[12]).

Step 2. Find all candidates for absolute extreme points of f that are on the boundary of the region. This part of the problem is significantly more complicated than the corresponding calculation in first semester calculus. If the boundary of the region has endpoints, then those endpoints certainly are candidates for absolute extreme points of f. In all cases the majority of the boundary of R consists of curves. Parameterize these boundary curves. (You might have to do this in more than one piece!) Look at f restricted to each of the boundary curves as a function of the parameter. Apply the first semester calculus technique to f restricted to each of the parameterized boundary curves. (You will understand the instructions for Step 2 once you see an example or two.)

Step 3. Once you have collected all of the candidates for the absolute extreme points of f on R, plug each candidate into f. Identify the absolute maximum and the absolute minimum.

Example 1. Find the absolute extreme points of $f(x,y) = 2x^2 - y^2 + 6y$ on $x^2 + y^2 \leq 16$.

Step 1. We calculate $f_x = 4x$ and $f_y = -2y + 6$. We see that f_x and f_y are both zero at $(0,3)$. **Save this point for use in Step 3.**

[12] In this course, you won't have to worry about this second option. Again, this is something you can worry about after the course is over, if you like.

The domain of f is everything inside or on the circle $x^2+y^2=16$. The boundary of the domain is $x^2+y^2=16$.

We could parameterize this circle in two pieces

- First piece: $y=\sqrt{16-x^2}$ for $-4\le x\le 4$ and
- Second piece: $y=-\sqrt{16-x^2}$ for $-4\le x\le 4$.

If we go this route, we have introduced the two end points $(-4,0)$ and $(4,0)$.

It turns out that I am rather lazy. If I can parameterize the boundary in one piece with one end point, I would prefer to do that rather than to parameterize the boundary with two pieces and two end points.

I parameterize the boundary as[13]

$$\begin{cases} x=4\cos\theta \\ y=4\sin\theta \end{cases}$$

with $0\le\theta\le 2\pi$. (I have one endpoint because $\theta=0$ and $\theta=2\pi$ both give the point $(4,0)$. **Save this point for use in Step 3.**) Let $f|_{\text{bdry}}$ represent "f restricted to the boundary". We see that

$$f|_{\text{bdry}}=2(4\cos\theta)^2-(4\sin\theta)^2+6(4\sin\theta).$$

In other words,

$$f|_{\text{bdry}}=32\cos^2\theta-16\sin^2\theta+24\sin\theta.$$

$$\begin{aligned}\frac{d}{d\theta}(f|_{\text{bdry}}) &= 64\cos\theta(-\sin\theta)-32\sin\theta\cos\theta+24\cos\theta \\ &= -96\cos\theta\sin\theta+24\cos\theta \\ &= -24\cos\theta(4\sin\theta-1)\end{aligned}$$

Thus, $\frac{d}{d\theta}(f|_{\text{bdry}})=0$ when $\cos\theta=0$ or $\sin\theta=\frac{1}{4}$. If $\cos\theta=0$, then θ is equal to $\frac{\pi}{2}$ or $\frac{3\pi}{2}$ and (x,y) is equal to $(0,4)$ or $(0,-4)$. If $\sin\theta=\frac{1}{4}$, then we draw the standard right triangles with OP $=1$, HYP $=4$, and ADJ $=\sqrt{15}$ or ADJ $=-\sqrt{15}$ to see that $\cos\theta$ is equal to $\frac{\sqrt{15}}{4}$ or $-\frac{\sqrt{15}}{4}$. (I will put a picture of these triangles at the end of this problem.) The corresponding points on the circle $x^2+y^2=16$ where $\frac{d}{d\theta}(f|_{\text{bdry}})=0$ are $(\sqrt{15},1)$ and $(-\sqrt{15},1)$ (because $x=4\cos\theta$ and $y=4\sin\theta$).

The full set of points on the circle $x^2+y^2=16$ where $\frac{d}{d\theta}(f|_{\text{bdry}})=0$ is

$$(0,4),\quad (0,-4),\quad (\sqrt{15},1),\quad \text{and}\quad (-\sqrt{15},1)$$

Save all four of these points for use in Step 3. We must consider the "endpoint" of the boundary $(4,0)$. We must consider the interior critical point $(0,3)$.

[13]Feel free to use the parameterization by $y=\sqrt{\dots}$ and $y=-\sqrt{\dots}$. You will get the same ultimate answer. You may prefer one parameterization more than the other.

Step 3. The absolute extreme points of f occur at some of these points we saved in Steps 1 or 2. We compute

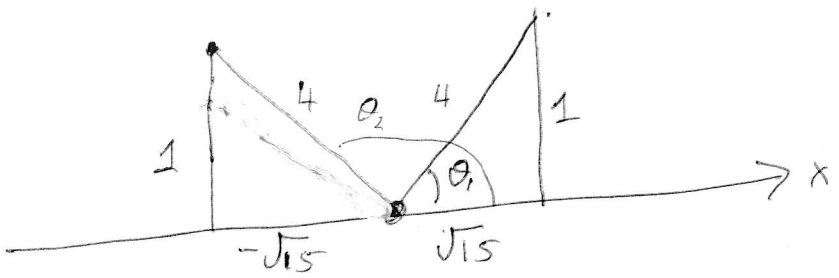
$$\begin{aligned} f(0,4) &= 2(0)^2 - (4)^2 + 6(4) = 8 \\ f(0,-4) &= 2(0)^2 - (-4)^2 + 6(-4) = -40 \\ f(\sqrt{15},1) &= 2(\sqrt{15})^2 - (1)^2 + 6(1) = 35 \\ f(-\sqrt{15},1) &= 2(-\sqrt{15})^2 - (1)^2 + 6(1) = 35 \\ f(4,0) &= 2(4)^2 - (0)^2 + 6(0) = 32 \\ f(0,3) &= 2(0)^2 - (3)^2 + 6(3) = 9 \end{aligned}$$

We conclude that

$(\sqrt{15},1,35)$ and $(-\sqrt{15},1,35)$ are the absolute maximum of $z = f(x,y)$ on $x^2+y^2 \leq 16$, and $(0,-4,-40)$ is the absolute minimum of $z = f(x,y)$ on $x^2+y^2 \leq 16$

I promised to draw a picture which shows how to calculate $\cos\theta$ if you know $\sin\theta = 1/4$. See the next page.

Suppose $\sin\theta = \frac{1}{4}$ and $0 \le \theta \le 2\pi$.
What is $\cos\theta$?

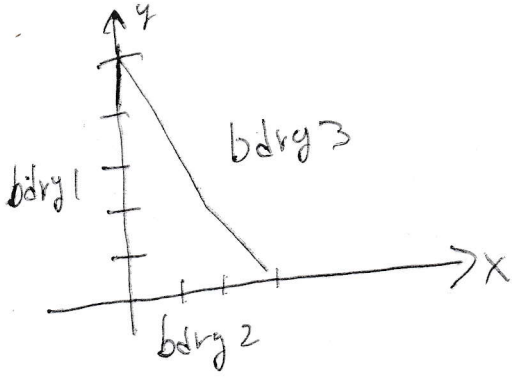


There are two choices for θ, namely θ_1 and θ_2

Observe that $\cos\theta_1 = \frac{ADJ}{HYP} = \frac{\sqrt{15}}{4}$

$\cos\theta_2 = \frac{ADJ}{HYP} = \frac{-\sqrt{15}}{4}$

The region for Example 2



Example 2. Find the absolute extreme points of $f(x,y) = 3xy - 6x - 3y + 7$ on the triangular region with vertices $(0,0)$, $(3,0)$, and $(0,5)$.

On the previous page we drew the domain R and we labeled the three parts of the boundary bdry1, bdry2, and bdry3. Of course the boundary has three endpoints: $(0,0)$, $(3,0)$, and $(0,5)$. **Save these three points for step 3.**

Step 1. We compute $f_x = 3y - 6$ and $f_y = 3x - 3$. We see that f_x and f_y are both zero at the point $(1,2)$. We verify that this point is in our triangular region;[14]so we **save this point for step 3.**

Step 2. We treat each piece of the boundary separately.

bdry 1. The y-axis is $x = 0$; thus bdry1 is $x = 0$ with $0 \leq y \leq 5$ and $f|_{\text{bdry1}} = -3y + 7$. We compute $\frac{d}{dy}(f|_{\text{bdry1}}) = -3$ and -3 is never zero. Thus, bdry1 does not contribute any points to Step 3 (other than the endpoints).

bdry 2. The x-axis is $y = 0$; thus bdry2 is $y = 0$ with $0 \leq x \leq 3$ and $f|_{\text{bdry2}} = -6x + 7$. We compute $\frac{d}{dx}(f|_{\text{bdry2}}) = -6$ and -6 is never zero. Thus, bdry2 does not contribute any points to Step 3 (other than the endpoints).

bdry 3. This part of the boundary is the line through $(0,5)$ and $(3,0)$. The line has slope $\frac{-5}{3}$. The equation of the line is $y - 5 = \frac{-5}{3}(x - 0)$ or $y = 5 - \frac{5}{3}x$ for $0 \leq x \leq 3$. Thus,

$$f|_{\text{bdry3}} = 3x(5 - \tfrac{5}{3}x) - 6x - 3(5 - \tfrac{5}{3}x) + 7.$$

In other words,

$$f|_{\text{bdry3}} = 14x - 5x^2 - 8.$$

We compute

$$\tfrac{d}{dx}(f|_{\text{bdry3}}) = 14 - 10x.$$

Thus, $\frac{d}{dx}(f|_{\text{bdry3}}) = 0$ when $x = \frac{7}{5}$. We then compute that $y = 5 - \frac{5}{3}\left(\frac{7}{5}\right) = \frac{8}{3}$. **Save the point $(\frac{7}{5}, \frac{8}{3})$ for step 3.**

We have calculated all candidates for the absolute extreme points. We found all interior critical points. We found all boundary critical points on all three pieces of the boundary. We found all boundary points on the boundary. We are ready to do step 3.

$$f(1,2) = 1$$
$$f(0,0) = 7$$
$$f(3,0) = -11$$
$$f(0,5) = -8$$
$$f(\tfrac{7}{5}, \tfrac{8}{3}) = \tfrac{9}{5}$$

[14]Actually, I don't verify that $(1,2)$ is in our region until I know that the equation of bdry3 is $y = 5 - \frac{5}{3}x$. At that point I happily see that $2 < \frac{10}{3} = 5 - \frac{5}{3}(1) = \text{bdry3}(1)$.

We conclude that

$(0,0,7)$ is the absolute maximum of $z = f(x,y)$ on the triangular region; and
$(3,0,-11)$ is the absolute minimum of $z = f(x,y)$ on the triangular region.

16.C. **Section 14.8. Lagrange multipliers.** One can use Lagrange multipliers to find the absolute extreme points of $z = f(x,y)$ on the level set $g(x,y) = k$.

Possibly $g(x,y) = k$ is (part of) the boundary of an absolute extreme point problem or maybe the Lagrange multiplier problem is a complete problem in its own right.

The idea is based on the fact that **GRADIENTS ARE PERPENDICULAR TO LEVEL SETS.**

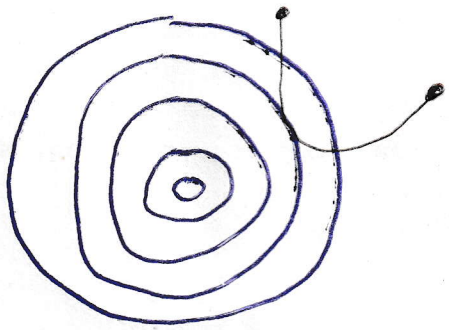
Imagine some level sets of $z = f(x,y)$. Now imagine the constraint $g(x,y) = k$ imposed on the collection of level sets for $z = f(x,y)$.

The absolute extreme points of $z = f(x,y)$ restricted to the constraint $g(x,y) = k$ occur either

1. at a point where $g(x,y) = k$ is tangent[15] to the level set of f or
2. at an endpoint of the constraint, or
3. at a non-smooth point on $g(x,y) = k$.

[15]If you are walking along $g(x,y) = k$ and cross a level set of f, then we are climbing or going downhill. The other option is that you are tangent to a level set of f. In this second option, you have reached the top or the bottom.

Some level sets of f are in blue



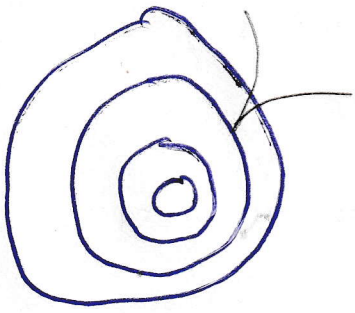
The constaint $g(x,y)=k$ is in black.

Observe that the absolute extreme points of $z=f(x,y)$ on the constraint $g(x,y)=k$ occurs at one of the following 3 types of points.

(1) A point (a,b) where $\vec{\nabla} f|_{(a,b)}$ is parallel to $\vec{\nabla} g|_{(a,b)}$ or

(2) An end point of $g(x,y)=k$ or

(3) A point (a,b) where $\vec{\nabla} g|_{(a,b)} = 0$



Example 16.1. Find the maximum of $f(x,y) = x^3y$ subject to the constraint $x^2+3y^2 = 48$.

We let $g(x,y) = x^2+3y^2$. Notice that the graph of $g(x,y) = 48$ is an ellipse; all points on the graph are smooth points and there are no end points. We look for all points $P = (x,y)$ with

$$\begin{cases} g(P) = 48 \\ \lambda \overrightarrow{\nabla} g|_P = \overrightarrow{\nabla} f|_P \end{cases}$$

for some number λ. We find all (x,y,λ) with

$$\begin{cases} x^2+3y^2 = 48 \\ \lambda(2x\overrightarrow{\boldsymbol{i}} + 6y\overrightarrow{\boldsymbol{j}}) = 3x^2y\overrightarrow{\boldsymbol{i}} + x^3\overrightarrow{\boldsymbol{j}} \end{cases}$$

We find all (x,y,λ) with

$$\begin{cases} x^2+3y^2 = 48 \\ \lambda 2x = 3x^2y \\ \lambda 6y = x^3 \end{cases}$$

The middle equation gives either $x = 0$ or $2\lambda = 3xy$.

If $x = 0$, then the bottom equation says that $y = 0$ or $\lambda = 0$. We see that $(x,y) = (0,0)$ does not satisfy the top equation. We are left with two cases. Either $x = 0$ and $\lambda = 0$ and $y = \pm 4$ or

$$\begin{cases} x \neq 0 \\ x^2+3y^2 = 48 \\ 2\lambda = 3xy \\ \lambda 6y = x^3 \end{cases}$$

The second case is equivalent to

$$\begin{cases} x \neq 0 \\ x^2+3y^2 = 48 \\ 2\lambda = 3xy \\ \frac{3xy}{2} 6y = x^3 \end{cases} \quad \text{or} \quad \begin{cases} x \neq 0 \\ x^2+3y^2 = 48 \\ 2\lambda = 3xy \\ 9y^2 = x^2 \end{cases} \quad \text{or} \quad \begin{cases} x \neq 0 \\ 12y^2 = 48 \\ 2\lambda = 3xy \\ 9y^2 = x^2 \end{cases}$$

Thus, $y^2 = 4$; hence $y = 2$ or $y = -2$. When $y = 2$, then $36 = x^2$ and when $y = -2$, then $36 = x^2$ The candidates for absolute max and absolute min are

$$(x,y) = (6,2), (-6,2), (6,-2), (-6,-2), (0,4), (0,-4).$$

We plug the candidates into f:

$$\begin{aligned} f(6,2) &= 2(216) \\ f(-6,2) &= -2(216) \\ f(6,-2) &= -2(216) \\ f(-6,-2) &= 2(216) \\ f(0,4) &= 0 \\ f(0,-4) &= 0. \end{aligned}$$

We conclude that

the maximum points of f on $x^2+3y^2=48$ are $(6,2,432)$ and $(-6,-2,432)$ and the minimum points of f on $x^2+3y^2=48$ are $(-6,2,-432)$ and $(6,-2,-432)$.

Example 16.2. Find the absolute extreme points of $f(x,y)=4x+6y+5$ subject to the constraint $x^2+3y^2=28$.

We use the method of Lagrange multipliers. Let $g(x,y)=x^2+3y^2$. Notice that the graph of $g(x,y)=28$ is an ellipse; consequently, the constraint is smooth and has no endpoints. The absolute extreme points of $z=f(x,y)$ on $g(x,y)=28$ at points P on $g(x,y)=28$ where $\overrightarrow{\nabla} f|_P$ is parallel to $\overrightarrow{\nabla} g|_P$. We find all such points.

We find x,y,λ with

$$\begin{cases} g(x,y)=28 \\ \lambda\overrightarrow{\nabla} f|_{(x,y)} = \overrightarrow{\nabla} g|_{(x,y)} \end{cases}$$

$$\begin{cases} x^2+3y^2=28 \\ \lambda(4\overrightarrow{i}+6\overrightarrow{j}) = (2x\overrightarrow{i}+6y\overrightarrow{j}) \end{cases}$$

$$\begin{cases} x^2+3y^2=28 \\ 4\lambda=2x \\ 6\lambda=6y \end{cases}$$

$$\begin{cases} x^2+3y^2=28 \\ 2\lambda=x \\ \lambda=y \end{cases}$$

$$\begin{cases} (2\lambda)^2+3\lambda^2=28 \\ 2\lambda=x \\ \lambda=y \end{cases}$$

$$\begin{cases} 7\lambda^2=28 \\ 2\lambda=x \\ \lambda=y \end{cases}$$

$$\begin{cases} \lambda = 2 \text{ or } -2 \\ 2\lambda = x \\ \lambda = y \end{cases}$$

When $\lambda = 2$, then $x = 4$ and $y = 2$. When $\lambda = -2$, then $x = -4$ and $y = -2$. Observe that

$$f(4,2) = 4(4) + 6(2) + 5 = 33 f(-4,-2) = 4(-4) + 6(-2) + 5 = \quad -23$$

We conclude that

$(4,2,33)$ is the absolute maximum of f on $g = 28$ and
$(-4,-2,-23)$ is the absolute minimum of f on $g = 28$.

17. Sections 15.1, 15.2, and 15.3. Double integrals.

The definite integral that you studied in first semester calculus, the double integral that we begin to study today, the triple integral that we study beginning with 15.5, and the line integral of Chapter 16 all are defined in more or less the same manner. (At this point, I am not writing about how to compute the things; I am writing about what they mean. There is no point computing them if you don't know what they mean. Furthermore, the applications are derived from the definitions of the objects and the reason you learn about these integrals is because they have applications.) At any rate, I will define the definite integral, the double integral, the triple integral, and the line integral fairly carefully.

I first define the definite integral. I hope that this definition looks somewhat familiar from first semester calculus.

The (verbose) **definition of the definite integral.** Let a and b be numbers with $a \leq b$ and let $f(x)$ be a function which is defined for $a \leq x \leq b$. We define the definite integral of f on the closed interval $[a,b]$; this object is denoted

$$\int_a^b f(x)\,dx.$$

The definition goes like this[16]. Partition the closed interval $[a,b]$ into n pieces (for some positive integer n); so

$$a = x_0 \leq x_1 \leq \cdots \leq x_{n-1} \leq x_n = b.$$

For each index i with $1 \leq i \leq n$, let $\Delta_i = x_i - x_{i-1}$ and pick an element x_i^* in the interval $[x_{i-1}, x_i]$. Multiply $f(x_i^*)$ times Δ_i. Add up all of the products

$$\sum_{i=1}^{n} f(x_i^*)\Delta_i.$$

Now take the limit as n goes to infinity and all Δ_i go to zero. If the limit exists, then the limit is called the definite integral of f on $[a,b]$ and is denoted

$$\int_a^b f(x)\,dx.$$

Possibly, some people would prefer that I rewrite the above discussion with fewer words. (I will, but I am just as happy with the verbose version.)

The (concise) **definition of the definite integral.** If f is a function which is defined for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx = \lim_{\text{all } \Delta_i \to 0} \sum_{i=1}^{n} f(x_i^*)\Delta_i$$

where $a = x_0 \leq x_1 \leq x_2 \leq \cdots \leq x_n = b$, $\Delta_i = x_i - x_{i-1}$, and $x_{i-1} \leq x_i^* \leq x_i$.

[16] By the way, the definite integral is a number.

First Application of the definite integral: AREA. If $f(x)$ is a continuous non-negative function defined for $a \leq x \leq b$, then

$$\int_a^b f(x)\,dx$$

is the area of the region[17] whose top is $y = f(x)$, bottom is the x-axis, left side is $x = a$, and right side is $x = b$.

Picture 1 on the next page is a picture of the first application of the definite integral.

Picture 2 on the next page tries to suggest why this application makes sense. Suppose someone wanted[18] the area of our shape. (This shape has a flat bottom, straight sides, and a curvy top.[19]) The natural thing to do is to say is "I'll pretend the region comes in pieces, I'll pretend each piece is a rectangle, I know how to find the area of a rectangle. That will be close enough." In Picture 2, we chopped the base into three pieces. Observe that

$$f(x_1^*)\Delta_1 + f(x_2^*)\Delta_2 + f(x_3^*)\Delta_3$$

is exactly equal to the area of the approximation. Of course, the approximation does not have exactly the same area as the official shape. One would get a better approximation by chopping the official shape into smaller pieces and using smaller bases. The exact area is the limit of the above process, if the limit exists.[20]

[17] One reason for studying double and triple integrals is that when one uses a double or triple integral one need not be nearly so fussy about how the region in 2-space or 3-space looks.

[18] The person wants the area for a practical reason. He or she wants to paint it, or tile it, or encrust with jewels, or whatever. The cost of the job will depend on the area of region.

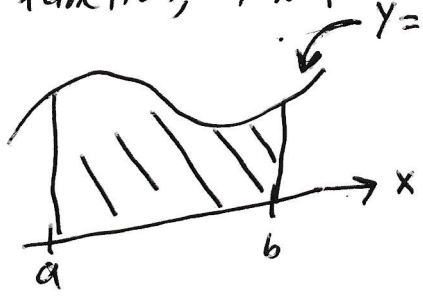
[19] One reason for studying double and triple integrals is that when one uses a double or triple integral one need not be nearly so fussy about how the region in 2-space or 3-space looks.

[20] The miracle is that the limit does exist and is easy to calculate. This is called the Fundamental Theorem of Calculus. You are quite familiar with how to calculate $\int_a^b f(x)\,dx$. (One finds the anti-derivative of f and plugs in the end points.) You are so used to calculating $\int_a^b f(x)\,dx$, that you might have forgotten what exactly you are calculating. This discussion is an attempt to refresh your memory.

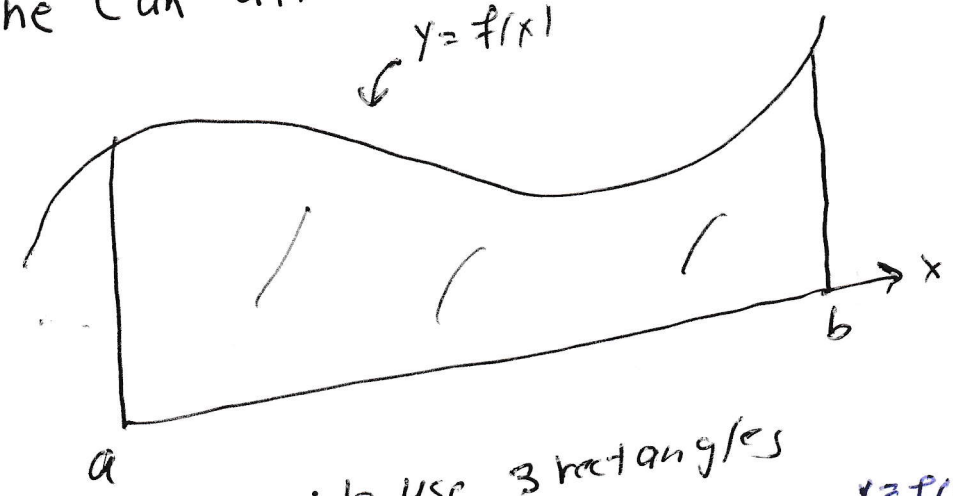
Application 1 of the definite integral

① If f is a continuous non-negative function, then

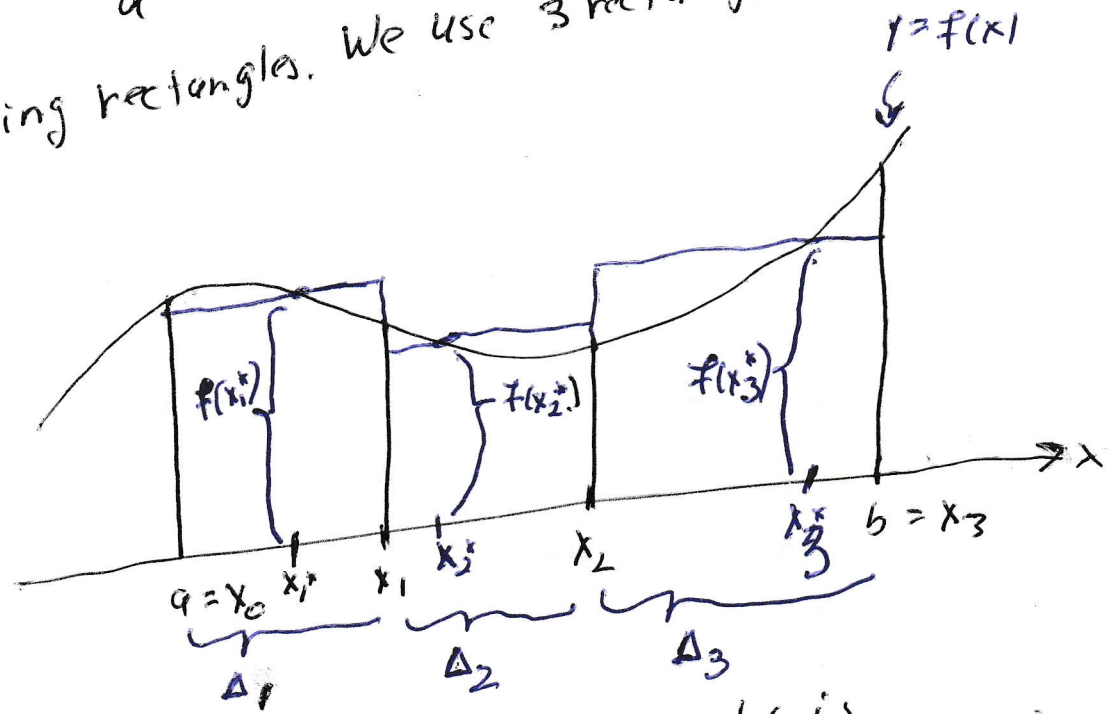
$$\int_a^b f(x)\,dx = \text{the area of}$$



② One can approximate the area of



using rectangles. We use 3 rectangles



the area inside the 3 rectangles is

$$f(x_1^*)\,\Delta_1 + f(x_2^*)\,\Delta_2 + f(x_3^*)\,\Delta_3$$

Second Application of the definite integral: WORK

Let $f(x)$ be a force which is aimed in a single direction; although $f(x)$ is not necessarily constant. The standard example of such a force is the force of a spring with one end held fixed. There is a picture of one such spring on the next page. The more I pull on the free end the more spring fights against me. I want[21] to calculate how much work is done as I stretch the spring from $x = a$ to $x = b$.

The relevant fact from Physics is that if **the force is constant and in single direction and the object is moved in the direction of the force, then**

Work is equal to force times distance.

In our problem, the force is not constant, so we chop the interval $[a,b]$ into small pieces. In each piece, we *pretend* that the force is constant. We find the work as the object is moved from one end of the small piece to the other end by the pretend constant force:

$$f(x_i^*)\Delta_i.$$

We add up all of these approximations of the work done. If we want a better approximation of the work done, we chop $[a,b]$ into smaller pieces. If we want to know the exact amount of work done, we take the limit. It is a miracle (called the Fundamental Theorem of Calculus) that the limit exists, is easy to compute, and can be computed using anti-derivatives.

At any rate we have the following fact.

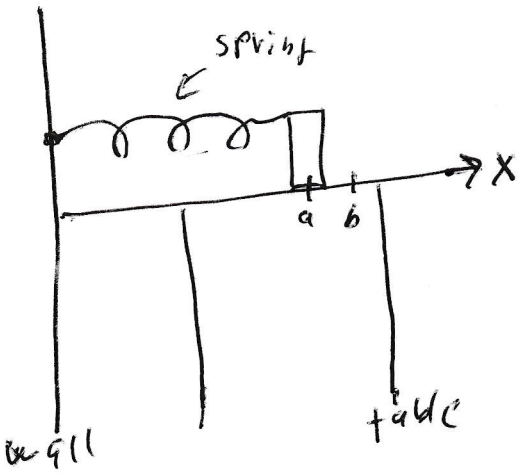
Fact If $f(x)$ is a force which is aimed in one direction then the work done as an object is moved in the direction of f from $x = a$ to $x = b$ is $\int_a^b f(x)\,dx$.

The next layer of sophistication is to allow the force's direction as well as its magnitude to change. To calculate the work done by such a force one uses a line integral. Line integrals are the subject of Chapter 16. There is a decent chance that the semester will end before we get there.

I promised a picture of a spring. See the next page.

[21] Of course, the reason I care is that the amount of energy I have to buy to do the job is directly calculated from the amount of work that is required to do the job. I want to know what the job will cost.

Spring



That is enough preamble. It is time to look at the topic of today's lecture. We define the double integral[22].

Double Integral. Let R be a region in the xy-plane and $f(x,y)$ be a function defined on R. We define the double integral of f over R, which is denoted[23]

$$\iint_R f(x,y)\,dA.$$

Use horizontal and vertical lines to partition R into rectangles. Number the rectangles from 1 to n. For each index i, with $1 \leq i \leq n$, let Δ_i be the area of rectangle i and pick a point (x_i^*, y_i^*) in rectangle i. Then the double integral of f over R is[24]

$$\iint_R f(x,y)\,dA = \lim_{\text{all } \Delta_i \to 0} \sum_{i=1}^{n} f(x_i^*, y_i^*)\Delta_i.$$

There is a picture of the "numbered rectangles" on the next page.[25] Notice that it does not matter what you do with rectangles that are part in R and part out of R. You can use them or not use them as you like. This error disappears as you take $\lim_{\text{all } \Delta_i \to 0}$.

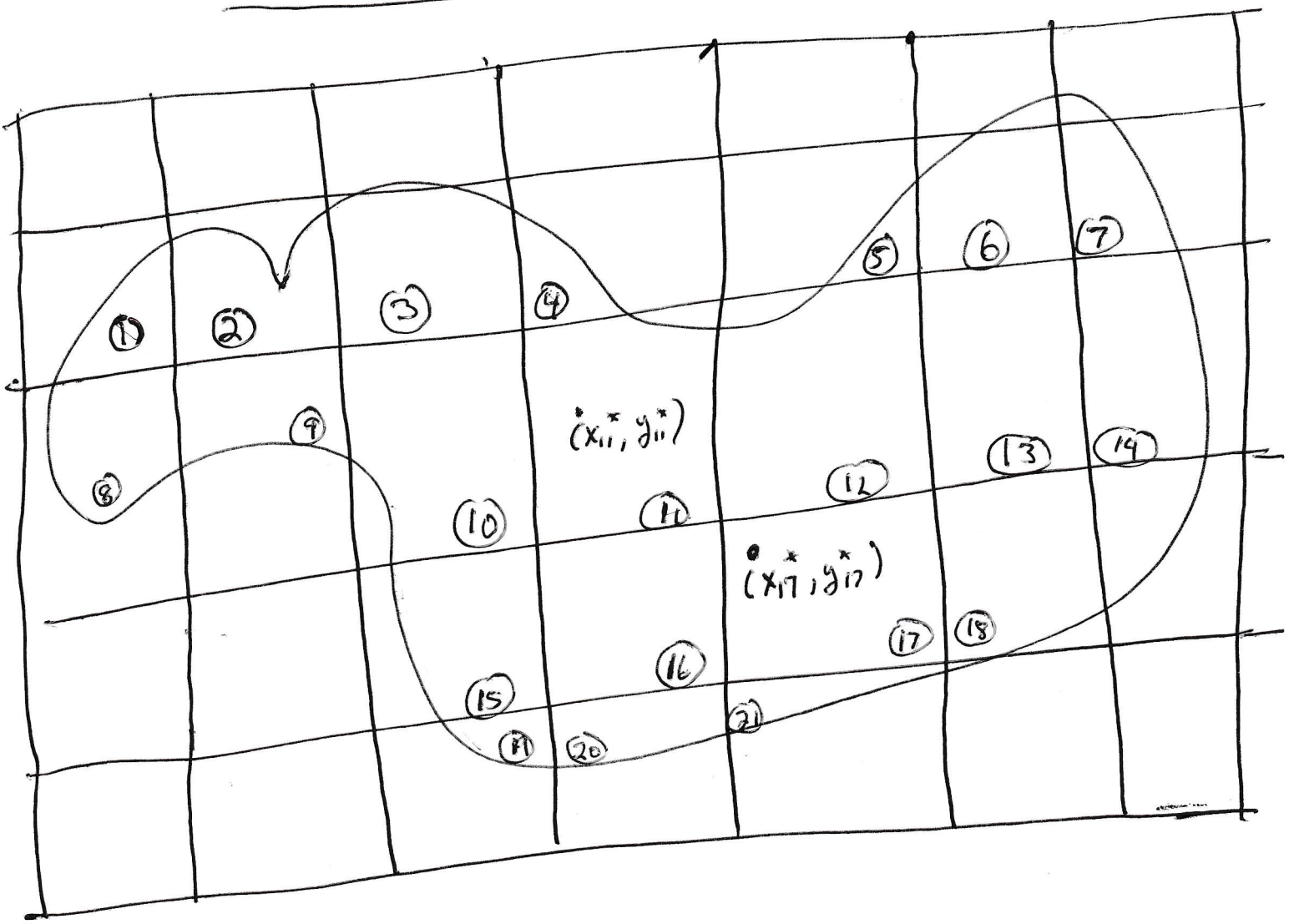
[22] By the way, the double integral is a number.

[23] The symbol "dA" indicates that we are considering a little piece of area.

[24] Notice that the double integral and the definite integral have essentially the same definition. As soon as you come to terms with the definition of the definite integral, then you also make sense of the double integral.

[25] I only marked points (x_i^*, y_i^*) in a few of the rectangles; but you get the idea: one picks a point (x_i^*, y_i^*) in each of the $n = 21$ rectangles.

Numbered Rectangles



In the first application we take $f(x,y) = 1$ for all (x,y) and since $f(x,y)$ is 1 and we multiply $f(x,y)$ times dA; we do not even have to write $1\,dA$ it is good enough to just write dA.

First Application of the double integral: AREA. If R is a region in the xy-plane, then

$$\iint_R dA = \text{the area of } R.$$

I hope you say, "YES, that is obvious." We have chopped R into small rectangles; computed $\sum \Delta_i$ (which is the area of an approximation of R) and then we took the limit as the approximation of R approaches R.

Of course, one might complain, "Big deal. I can use the definite integral to compute area. Why should I bother with this double integral?" The answer is clear. When one sets up a definite integral to calculate area, one must be very fussy: this is the top, this is the bottom, these are the sides. There is no fuss to write:
"If R is a region in the xy-plane, then

$$\iint_R dA = \text{the area of } R.\text{"}$$

(On the other hand, one has to examine R before one can calculate $\iint_R dA$.)

Second Application of the double integral: Volume. If S is a solid in 3-space with base equal to R, which is a region in the xy-plane, top given by $z = t(x,y)$ on or above the xy-plane, and straight sides, then

$$\iint_R t(x,y)dA = \text{the volume of } S.$$

I put a picture of the second application on the next page.

I hope that it is clear why the second application works. We have partitioned the base of S into n rectangles for some number n. We are approximating the solid S by a n three-dimensional blocks. The i^{th} block has a rectangular base with area Δ_i and has height $t(x_i^*, y_i^*)$; so the volume of the i^{th} block is

$$t(x_i^*, y_i^*)\Delta_i.$$

The exact volume of the approximation is

$$\sum_{i=1}^{n} t(x_i^*, y_i^*)\Delta_i.$$

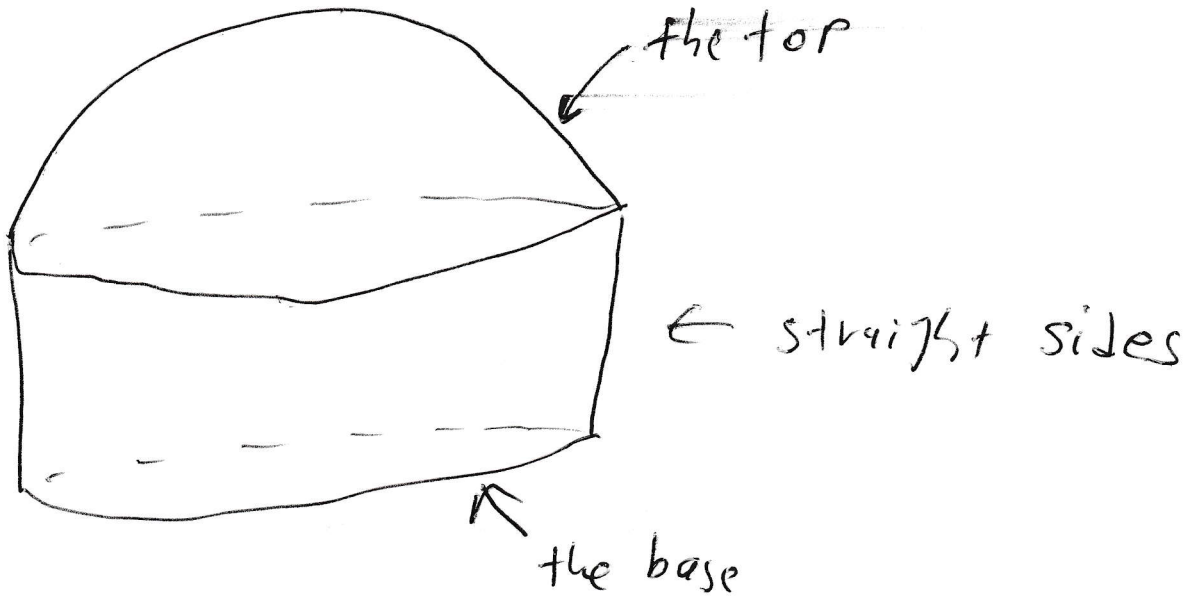
To get a better approximation of S, use more blocks, making each base smaller. The limit of this process (if it exists) is the volume of S.

I promised a picture of the second application of the double integral. This picture is on the next page.

The Second Application of the double Integral.

Let S be a solid with base equal to R, which is a region in the xy-plane, top given by $z = f(x,y)$, and straight sides, then

$$\iint_{\text{base}} f(x,y)\,dA = \text{the volume of } S$$



Third Application of the double integral: Volume, but less fussy. Let S be a solid in 3-space with top given by $z = t(x,y)$ and bottom given by $z = b(x,y)$. Suppose that R is a region that is parallel to the xy-plane and is a representative cross section of S. Then

$$\iint_R \Big(t(x,y) - b(x,y)\Big)\,dA = \text{the volume of } S.$$

- I put a picture of the third application of the double integral on the next page.
- Of course, the second application is the special case of the third application with $b(x,y) = 0$ for all (x,y) and R is IN the xy-plane not merely parallel to the xy-plane. Once you make sense of the third application, it is not necessary to also store the second application in your head.
- The third application makes sense because we approximate the solid with n three-dimensional rectangular blocks. We have partitioned the cross section R of S into n rectangles, for some n. The area of the i^{th}-cross sectional rectangle is Δ_i and the height of the i^{th} block is the z-coordinate at the top minus the z-coordinate at the bottom:

  $$t(x_i^*, y_i^*) - b(x_i^*, y_i^*),$$

  for some (x_i^*, y_i^*) in the i^{th} rectangle. The volume of the i^{th}-block is

  $$\Big(t(x_i^*, y_i^*) - b(x_i^*, y_i^*)\Big)\Delta_i.$$

  The volume of the approximation is exactly

  $$\sum_{i=1}^{n} \Big(t(x_i^*, y_i^*) - b(x_i^*, y_i^*)\Big)\Delta_i.$$

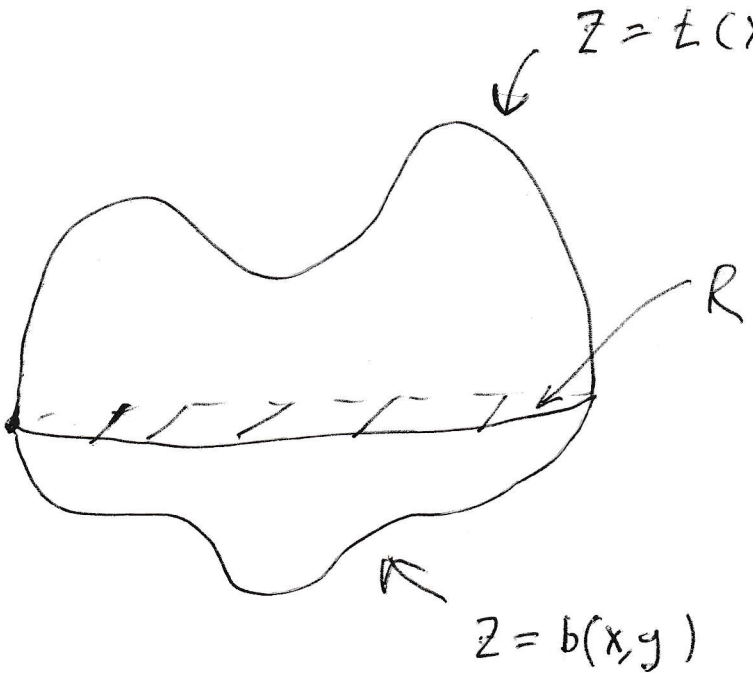
  To get a better approximation of S, use more blocks, making each Δ_i smaller. The limit of this process (if it exists) is the volume of S.
- If the third application of the double integral is still too fussy for your taste, soon we will have triple integrals and we will write "If S is a solid in 3-space, then $\iiint_S dV$ is the volume of S." This looks like a big deal, but we will still have to understand S in order to calculate $\iiint_S dV$.

The next page contains the promised picture of Application Three of the double integral.

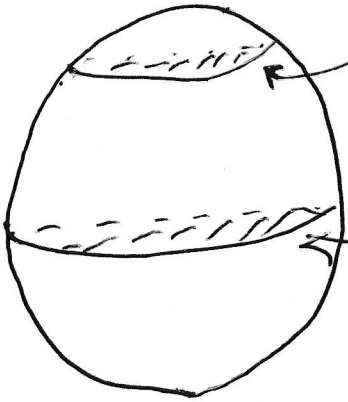
The third Appliration of the double integral

Let S be a solid with top given by $z = t(x,y)$ and bottom given by $z = b(x,y)$. Suppose R is a region that is parallel to the xy plane and is a representative cross section of S. Then

$$\iint_R \left(t(x,y) - b(x,y)\right) dA = \text{The Volume of } S.$$



If S were a sphere, then



R should be the fattest cross section.

How to compute double integrals.

Let R a region in the xy-plane. We fill R using either vertical lines or horizontal lines.

If we fill R using vertical lines, then

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{b(x)}^{t(x)} f(x,y)dy \right) dx. \tag{17.0.1}$$

Look at the next page to see that the vertical line $x = a$ is the left side of R, the vertical line $x = b$ is the right side of R, the curve $y = t(x)$ is the top of R and the curve $y = b(x)$ is the bottom of R. The integral on the left side of (17.0.1) is called a double integral; the integral on the rights side of (17.0.1) is called an iterated integral. When one computes the iterated integral in (17.0.1), one computes the inner integral first; one holds x constant and integrates the inner integral with respect to y; then one plugs $t(x)$ and $b(x)$ in for y. At this point, one is left with an ordinary definite integral of the form

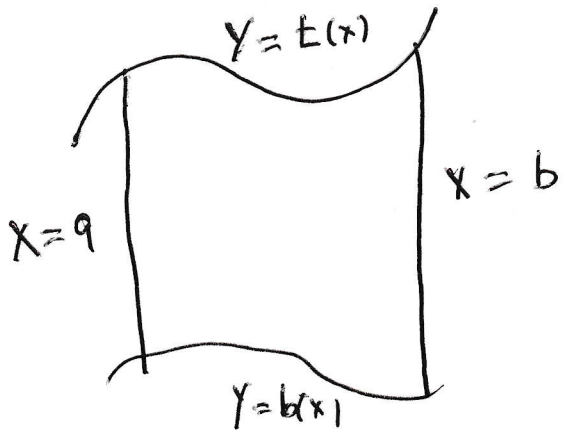
$$\int_a^b \text{ some function of } x\,dx,$$

which one computes to get the answer, which is a number.

Everything looks different if one fills R using horizontal lines. We will do that after the picture on the next page.

Fill R using vertical lines

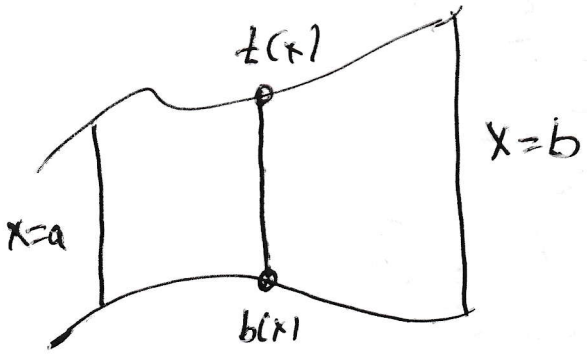
If R looks like:



then

$$\iint_R f(x,y)\,dA = \int_a^b \left(\int_{b(x)}^{t(x)} f(x,y)\,dy \right) dx.$$

One thinks: For each fixed x, with $a \le x \le b$, then y goes from the littlest possible value for that x (i.e. $b(x)$) to the biggest possible value for that x (i.e. $t(x)$)



Let R a region in the xy-plane. If we fill R in using horizontal lines, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{\ell(y)}^{r(y)} f(x,y)dx \right) dy. \tag{17.0.2}$$

Look at the picture on the next page to see that the horizontal line $y = c$ is the bottom of R, the horizontal line $y = d$ is the top of R, $x = \ell(y)$ is the left side of R, and $x = r(y)$ is the right side of R.

When one computes the iterated integral in (17.0.2), one computes the inner integral first; one holds y constant and integrates the inner integral with respect to x; then one plugs $r(y)$ and $\ell(y)$ in for x. At this point, one is left with an ordinary definite integral of the form

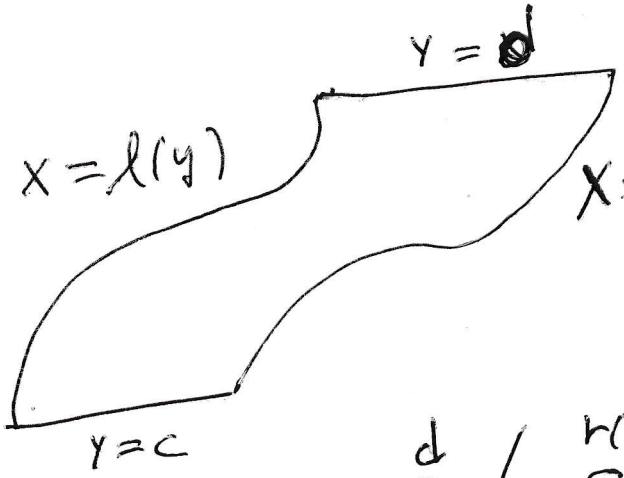
$$\int_c^d \text{ some function of } y\ dy,$$

which one computes to get the answer, which is a number.

The next page is the picture which is associated with filling the region R using horizontal lines.

Fill R using horizontal lines

If R looks like

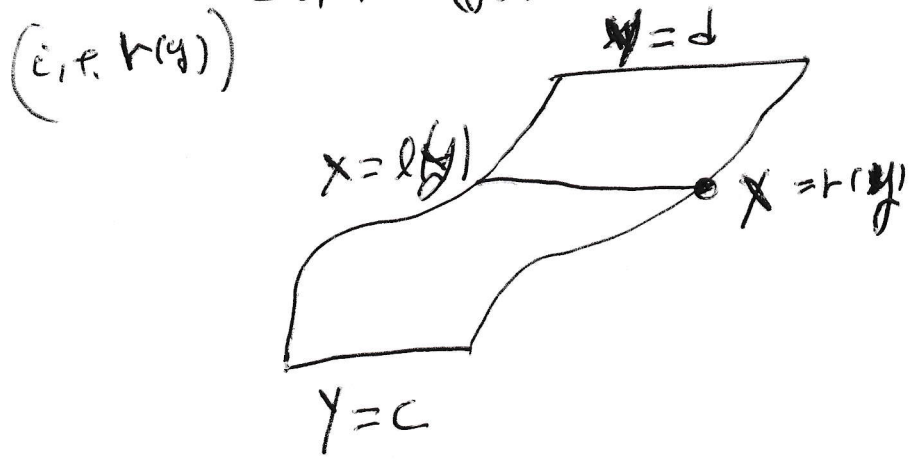


, then

$$\iint_R f(x,y)\,dA = \int_c^d \left(\int_{l(y)}^{r(y)} f(x,y)\,dx \right) dy.$$

One thinks: For each fixed y with $c \le y \le d$,

x goes from the littest possible value for that y (i.e. $l(y)$) to the biggest possible value for that y (i.e. $r(y)$)



I have many comments.

- Of course, in practice you have to decide if you want to fill R using vertical lines or horizontal lines.
- You might have to partition R into pieces and then attack each piece separately.
- Sometimes, there is an advantage to using one orientation rather than the other orientation because you can do the problem in one piece rather than several pieces.
- Sometimes, there is an advantage to using one orientation rather than the other orientation because you can actually do one of the integrals, but you are not able to do the other integral!
- **If there are variables on your outer integral, then your answer is wrong!**
- If all four limits on your integral signs are numbers, then you are integrating over a rectangle. If you are supposed to integrate over a rectangular, that is great. But if R is more complicated than a rectangle, then you have made a mistake.
- **In all cases, you have to sketch R.** It is impossible to set up a double integral correctly unless you have a sketch of R. You have to look at R and say, "I will fill R with vertical lines." or "I will fill R with horizontal lines."
- In first semester calculus you might have found area using both

$$\int_a^b f(x)dx \quad \text{and} \quad \int_c^d g(y)dy.$$

  The game we play with filling R with either vertical or horizontal lines is very similar to game of deciding if one wants to find the area of a region using $\int dx$ or $\int dy$. If you were good at it before. You will be good at it now. If you weren't any good at it before, well now you have another chance.
- Nobody actually puts parentheses in the iterated integral $\int_c^d \left(\int_{\ell(y)}^{r(y)} f(x,y)dx \right) dy$. Everyone simply writes $\int_c^d \int_{\ell(y)}^{r(y)} f(x,y)dxdy$

Example. The is problem 46 in 15.2. Integrate $\int_0^{\ln 2} \int_{e^y}^2 dxdy$ and do the same problem with the order of integration reversed.[26]

[26]There are times that one can not do the original integral; but one can do the integral after the order of integration is reversed. This is not a problem like that. We are asked to exchange the order of integration to practice the technique so that we can do it when we need it. But as a bonus, we get to do the problem a completely different way. If we get the same answer both times, there is a good chance we did both problems correctly. If we don't get the same answer both times, then it is absolutely certain that at least one of our answers is wrong. There is another reason to learn how to reverse the order of integration: If you can reverse the order of integration, then you have mastered the technique. You will be able to do any problem you are asked to do. On the other hand, if you are unable to reverse the order of integration, then your understanding of double integrals is very shallow; you are not likely to be able to do a typical problem.

First of all,

$$\int_0^{\ln 2}\int_{e^y}^{2} dx\,dy = \int_0^{\ln 2} x\Big|_{e^y}^{2} dy = \int_0^{\ln 2}(2-e^y)dy = (2y-e^y)\Big|_0^{\ln 2}$$
$$= (2\ln 2 - e^{\ln 2}) - (2(0)-e^0) = 2\ln 2 - 2 + 1 = \boxed{2\ln 2 - 1}.$$

To reverse the order of integration we have to figure out what we have integrated over. For each fixed y with $0 \le y \le \ln 2$, our x has gone from $x = e^y$ to $x = 2$. We graph $x = e^y$ (which is the same as $\ln x = y$) and $x = 2$ (which is a vertical line) Of course, $y = \ln x$ passes through the point $(1,0)$, has a vertical asymptote of the negative y-axis, and grows very slowly when $1 < x$. The two graphs intersect at $(2, \ln 2)$. Study the next page to see that

$$\int_0^{\ln 2}\int_{e^y}^{2} dx\,dy = \int_1^{2}\int_0^{\ln x} dy\,dx.$$

Be sure to notice that you must draw the graph in order to reverse the order of integration.

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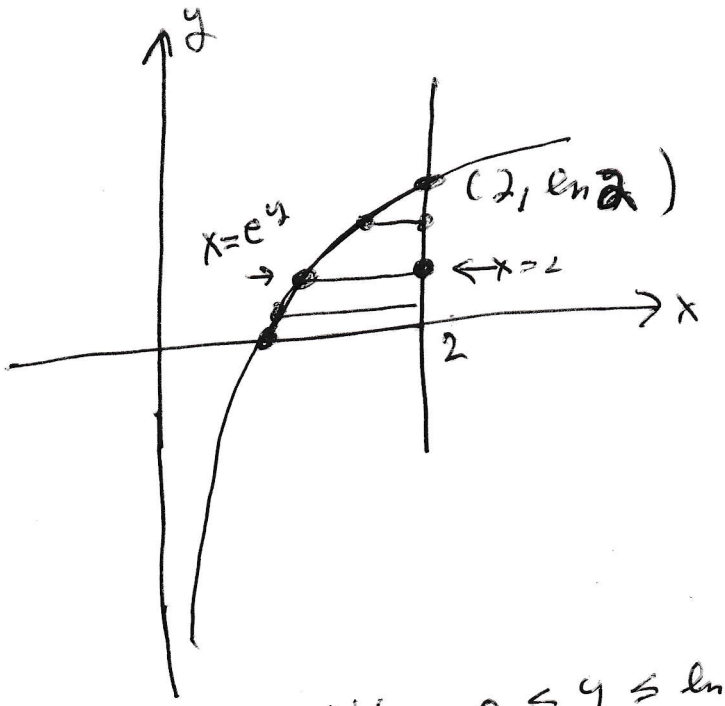
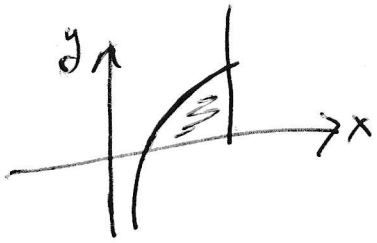
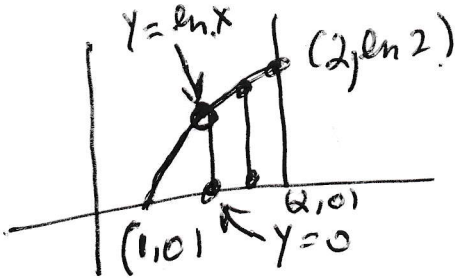


For y with $0 \le y \le \ln 2$ our region goes from $x=e^y$ to $x=2$.

Clearly our region is



We can fill this region with vertical lines



for each fixed x with $1 \le x \le 2$

y goes from 0 to $y = \ln x$

$$\int_0^{\ln 2} \int_{e^y}^{2} dx\, dy = \int_1^2 \int_0^{\ln x} dy\, dx$$

We finish the problem by computing

$$\int_1^2 \int_0^{\ln x} dy\,dx = \int_1^2 y\Big|_0^{\ln x} dx$$
$$= \int_1^2 \ln x\,dx$$

Use integration by parts to integrate $\ln x$. Recall integration by parts:

$$\int u dv = uv - \int v du.$$

Take $u = \ln x$ and $dv = dx$. Calculate $du = \frac{1}{x}dx$ and $v = x$. Thus

$$\int \ln x\,dx = x\ln x - \int dx = x\ln x - x + C.$$

By the way, this is correct. Indeed,

$$\tfrac{d}{dx}(x\ln x - x) = x(\tfrac{1}{x}) + \ln x - 1 = \ln x.$$

Now we can finish the problem

$$\int_1^2 \int_0^{\ln x} dy\,dx = \int_1^2 \ln x\,dx = (x\ln x - x)\Big|_1^2 = 2\ln 2 - 2 - (0-1) = \boxed{2\ln 2 - 1.}$$

Woo Hoo! This is the same answer as we got when we did the problem the other way.

Example. The is problem 66 in 15.2. Find the volume of the solid that is bounded above by the cylinder $z = x^2$ and bounded below by the region enclosed by the parabola $y = 2 - x^2$ and the line $y = x$ in the xy-plane.

I draw the top and the bottom separately; although it is not necessary to draw the top. Use Application 2 of the double integral. The volume is

$$\iint_{\text{base}} \text{top}\,dA = \iint_{\text{base}} x^2\,dA.$$

It is necessary to draw the base before proceeding. (I also drew the top; but I did not have to draw it. I only drew it for my own amusement.)

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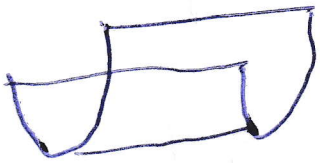
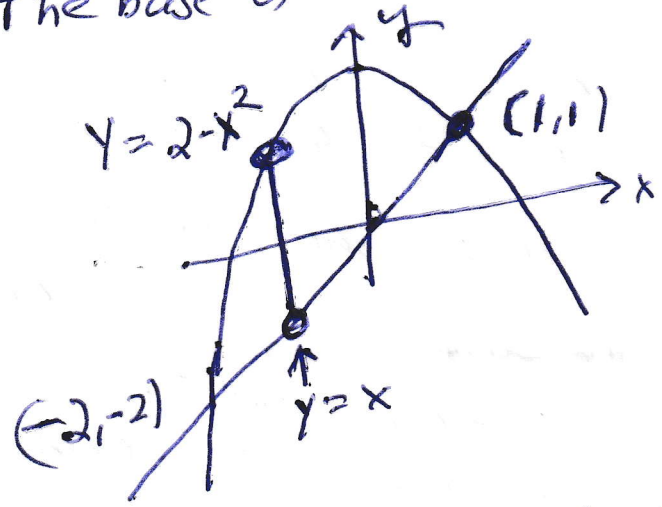
The top is $z = x^2$, which looks like:



I drew $z = x^2$ in the xz plane and then I drew the same picture for each y.

The base is



The intersection is found by solving $y = 2 - x^2$ and $y = x$ simultaneously: $x = 2 - x^2$ or $x^2 + x - 2 = 0$

$(x+2)(x-1) = 0$

$x = -2$ or $x = 1$

The parabola is $y = x^2$ turned upside down and lifted by 2.

If you use horizontal lines to fill the region, then you need two double integrals. So we fill the region with vertical lines

$$\text{The vol} = \int_{-2}^{1} \int_{x}^{2-x^2} x^2 \, dy \, dx.$$

Now we can finish the problem. The volume is

$$\int_{-2}^{1}\int_{x}^{2-x^2} x^2\,dy\,dx = \int_{-2}^{1} x^2 y\Big|_{x}^{2-x^2}\,dx = \int_{-2}^{1}\left(x^2(2-x^2)-x^2x\right)dx$$
$$= \int_{-2}^{1}(2x^2-x^4-x^3)\,dx = \left(\frac{2x^3}{3}-\frac{x^5}{5}-\frac{x^4}{4}\right)\Big|_{-2}^{1}$$
$$= \boxed{\left(\frac{2}{3}-\frac{1}{5}-\frac{1}{4}\right)-\left(\frac{2(-8)}{3}-\frac{-32}{5}-\frac{16}{4}\right)}$$

Example. This problem is 15.2, number 68. Find the volume of the solid in the first octant[27] which is bounded by the coordinate planes, the cylinder $x^2+y^2=4$, and the plane $z+y=3$. I think of the problem as a transition problem from sections 15.1, 15.2, and 15.3 to section 15.4. It is not hard to set up the double integral and it is "possible" to do the integral; but in order to do the integral, one has to use trig substitution. On the other hand, it is very easy to do the integral using polar coordinates; so that is what we will do. It turns out that polar coordinates is the topic of 15.4. We will do a sneak preview in this problem.

We get started in rectangular coordinates. I want you to see where the difficulty arises.

I drew the solid on a near by page. We can use Application 2 of the double integral. The volume is equal to

$$\iint_{\text{base}} \text{top}\,dA.$$

Look at the picture. We fill the base using horizontal lines and see that the volume is equal to

$$\int_0^2\int_0^{\sqrt{4-y^2}}(3-y)\,dx\,dy = \int_0^2(3-y)x\Big|_0^{\sqrt{4-y^2}}\,dy = \int_0^2(3-y)\sqrt{4-y^2}\,dy$$
$$= \int_0^2\left(3\sqrt{4-y^2}-y\sqrt{4-y^2}\right)dy.$$

THERE IS NO DIFFICULTY in finding

$$\int -y\sqrt{4-y^2}\,dy.$$

Just use ordinary substitution. Let $u=4-y^2$; so, $du=-2ydy$. (So $\frac{1}{2}du=-ydy$. I can put the $\frac{1}{2}$ where ever I want.) Thus

$$\int -y\sqrt{4-y^2}\,dy = \tfrac{1}{2}\textstyle\int\sqrt{u}du+C = \tfrac{1}{2}\tfrac{2}{3}u^{3/2}+C = \tfrac{1}{3}(4-y^2)^{3/2}+C.$$

By the way, this is correct because

$$\tfrac{d}{dy}\left(\tfrac{1}{3}(4-y^2)^{3/2}\right) = \tfrac{1}{3}\tfrac{3}{2}(-2y)(4-y^2)^{1/2} = -y\sqrt{4-y^2},$$

[27]The first octant in 3-space is the one eighth of 3-space where the x, y, and z coordinates are all non-negative.

as desired. Thus, the volume is

$$\int_0^2 3\sqrt{4-y^2}dy + \frac{1}{3}(4-y^2)^{3/2}\Big|_0^2.$$

Unfortunately, one must use Trig substitution to compute $\int \sqrt{4-y^2}dy$. I propose that we avoid doing Trig substitution and we switch the problem to polar coordinates instead. In polar coordinates, $x = r\cos\theta$ and $y = r\sin\theta$. We fill the base using straight lines which emanate from the origin. For each fixed angle θ, with $0 \le \theta \le \frac{\pi}{2}$, r goes from 0 to 2. In polar coordinates, the base is

$$\int_0^{\pi/2}\int_0^2.$$

(See the picture on the next page.) Of course, $3-y$ becomes $3-r\sin\theta$. **This one is not obvious:**

$$\boxed{dxdy \quad \text{becomes} \quad rdrd\theta.}$$

The reason is when one does a change of variables in an integral one has to account for the chain rule. We are very used to $du = \frac{du}{dx}dx$ when one only has one old variable and one new variable. When we switch from rectangular coordinates to polar coordinates, we must deal with four partial derivatives. One puts them in a matrix; takes the determinant; and then takes the absolute value. The upshot is

$$\boxed{dxdy \quad \text{becomes} \quad rdrd\theta.}$$

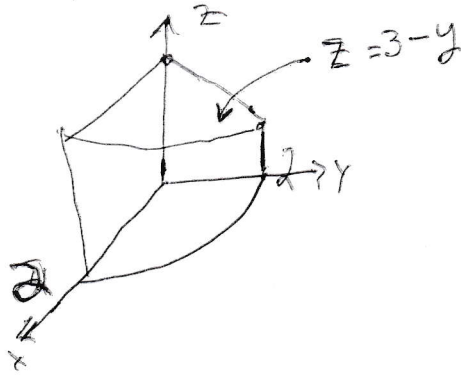
If that was too vague, we will do it again in 15.7. The general case is in 15.8 (but we probably will not cover it.) At any rate, the volume is

$$\int_0^{\pi/2}\int_0^2 (3-r\sin\theta)rdrd\theta = \int_0^{\pi/2}\int_0^2 (3r-r^2\sin\theta)\,drd\theta$$

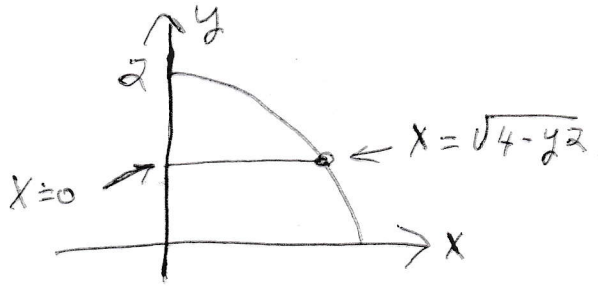
$$= \int_0^{\pi/2}\left(\tfrac{3r^2}{2} - \tfrac{r^3}{3}\sin\theta\right)\Big|_0^2 d\theta = \int_0^{\pi/2}(6-\tfrac{8}{3}\sin\theta)\,d\theta = \left(6\theta + \tfrac{8}{3}\cos\theta\right)\Big|_0^{\pi/2} = \boxed{3\pi - \frac{8}{3}.}$$

All pictures are on the next page. There is a picture of the solid, a picture of the base filled with horizontal lines, and a picture of the base filled with radial lines.

15.2-60



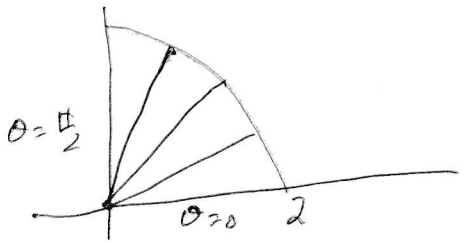
The solid



base

$$\text{Volume} = \int_0^2 \int_0^{\sqrt{4-y^2}} (3-y)\, dx\, dy$$

In Polar Coordinates the base is



For each fixed θ with $0 \le \theta \le \frac{\pi}{2}$
r goes from 0 to $r = 2$

$$\text{Vol} = \int_0^{\frac{\pi}{2}} \int_0^2 (3 - r\sin\theta)\, r\, dr\, d\theta$$

Note $dx\,dy$ becomes $r\,dr\,d\theta$

18. **Section 15.4. Double integrals in polar coordinates.**

We did problem 60 from Section 15.2 as the last example in Sections 15.1–15.3. To do the integral in rectangular coordinates we would have had to use trig substitution. We decided to use polar coordinates instead. The point is that **by switching to polar coordinates, we got an EASIER problem.** Sometimes, actually, it is **impossible** to do an integral in rectangular coordinates, but fairly routine to do the integral in polar coordinates. To integrate over a region in polar coordinates one fills the region with line segments that live on lines passing through the origin.

The region in question has four boundaries:

$$\theta = \theta_0, \quad \theta = \theta_1, \quad r = f_1(\theta), \quad \text{and} \quad r = f_2(\theta),$$

where $\theta_0 \leq \theta_1$ are fixed numbers and $f_1(\theta) \leq f_2(\theta)$ are functions. There is a big picture on the next page. One thinks, "For each fixed θ with θ between θ_0 and θ_1, r goes from the littlest value of r for that angle θ, namely $f_1(\theta)$, to the biggest value of r for that angle θ, namely $f_2(\theta)$. (Notice that when θ is fixed and one draws r from $f_1(\theta)$ to $f_2(\theta)$, then one is drawing at a line segment that lives on a line which passes through the origin.)

There is no trouble converting x and y into polar coordinates:

$$x = r\cos\theta \quad \text{and} \quad y = r\sin\theta.$$

There is one other issue, namely what does $dxdy$ (or $dydx$) become. The answer is

$$\boxed{dxdy \text{ (or } dydx\text{) becomes } rdrd\theta}.$$

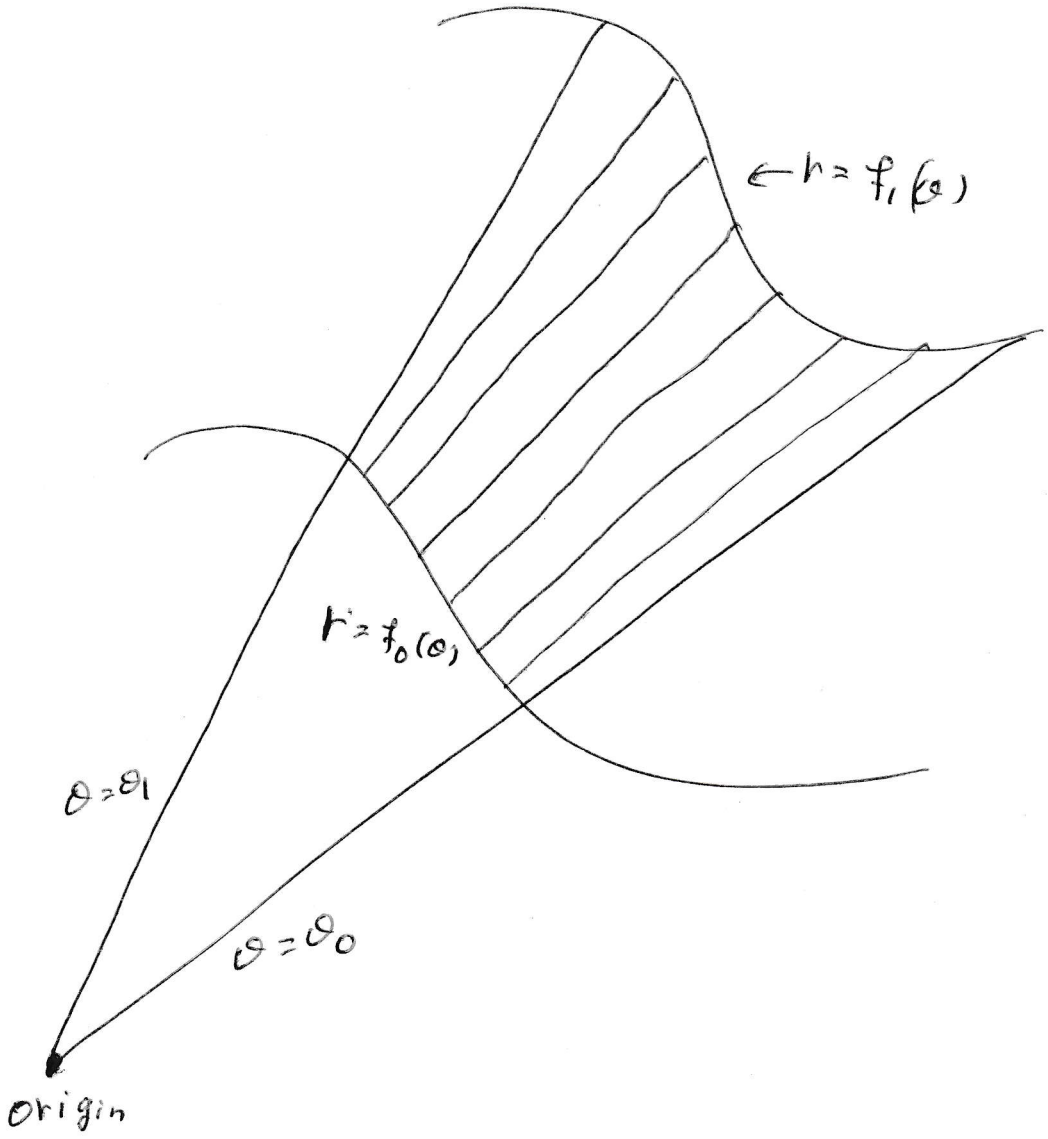
If you are satisfied, fine. If you are grumbling, "Where did that r come from?", then I will give an answer. I do not claim that I am giving the complete answer, but I hope that I give enough that the skeptical students think, "Well that is plausible."

The next page is a pretty picture. We continue with the discussion of

$$dxdy = rdrd\theta$$

after the picture.

Integrate in Polar Coordinates



Integrate over the shaded region using

$$\int_{\theta_0}^{\theta_1} \int_{f_0(\theta)}^{f_1(\theta)} r \, dr \, d\theta$$

I am giving some context for the fact $dxdy = rdrd\theta$. One does change of variables when one does the single integral. The process is called substitution. If one makes a substitution, $u = u(x)$, then one also replaces dx using the formula $du = \frac{du}{dx}dx$. If we start with an integral in terms of x and y and we change to new variables u and v by way of $x = x(u,v)$ and $y = y(u,v)$, then there are 4 partial derivatives to think about. We put the four partial derivatives into a matrix, we take the determinant, then we take the absolute value:

$$dxdy = \left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| dudv. \tag{18.0.1}$$

In particular, when $x = r\cos\theta$ and $y = r\sin\theta$, then formula (18.0.1) becomes

$$\begin{aligned} dxdy &= \left|\begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix}\right| drd\theta = \left|\begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}\right| drd\theta = |r\cos^2\theta + r\sin^2\theta| drd\theta \\ &= |r(\cos^2\theta + \sin^2\theta)| drd\theta = |r| drd\theta = rdrd\theta. \end{aligned}$$

We use the version

$$dxdy = rdrd\theta$$

here. In section 15.7 we use the version

$$dxdydz = \rho^2 \sin\phi d\rho d\phi d\theta.$$

for the switch from rectangular coordinates to spherical coordinates for triple integrals. Equation (18.0.1) is the content of section 15.8 and is very useful.

Now we do three examples.

Example 1. Compute $\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx$.

None of us can integrate $\int e^{y^2}\,dy$! So this is a trick problem! We recognize that $x^2+y^2=r^2$ in polar coordinates. We also recognize that $y=\sqrt{1-x^2}$ is part of the circle $x^2+y^2=1$. (Notice that $y=\sqrt{1-x^2}$ is the part of the circle $x^2+y^2=1$ where y is positive. Thus, $y=\sqrt{1-x^2}$ is the upper semi-circle.) Lets switch to polar coordinates.

We draw the region on the next page. For each fixed x, with $-1\le x\le 0$, y goes from the x-axis (also known as $y=0$) to the upper part of the circle $x^2+y^2=1$. When we view this region in polar coordinates, we think, for each fixed θ between $\theta=\frac{\pi}{2}$ and $\theta=\pi$, r goes from 0 to 1.

$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx=\int_{\pi/2}^{\pi}\int_{0}^{1} re^{r^2}\,dr\,d\theta$$

To integrate $\int re^{r^2}dr$ you probably let $u=r^2$. In this case, $du=2rdr$ and

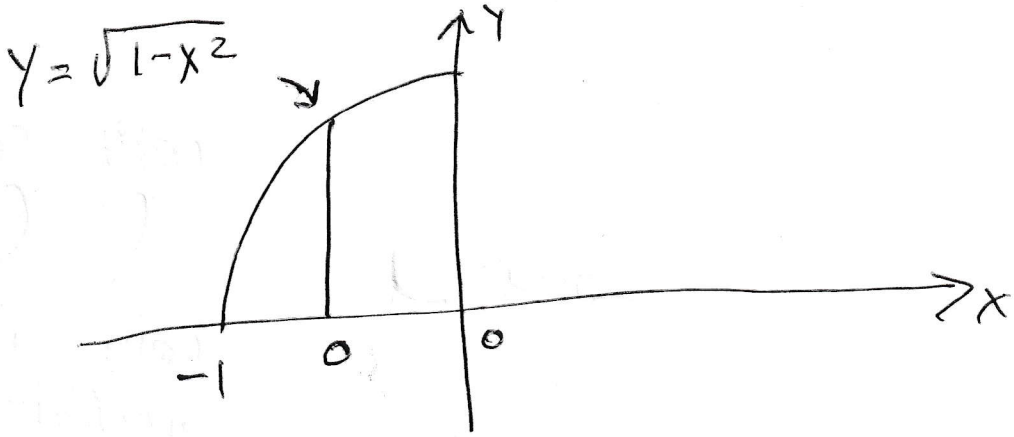
$$\int re^{r^2}dr=(1/2)\int e^u du=(1/2)e^u+C=(1/2)e^{r^2}+C.$$

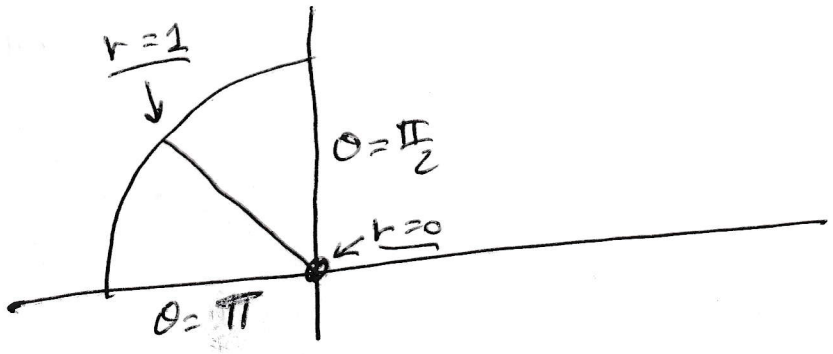
(Of course, one can check this in one's head.) The original integral is

$$\frac{1}{2}\int_{\pi/2}^{\pi} e^{r^2}\Big|_0^1 d\theta=\frac{1}{2}\int_{\pi/2}^{\pi}(e-1)\,d\theta=\frac{(e-1)}{2}\theta\Big|_{\pi/2}^{\pi}=\frac{(e-1)}{2}\left(\pi-\frac{\pi}{2}\right)=\boxed{\frac{(e-1)\pi}{4}.}$$

The picture is on the next page.

15.4 First Example





$$\int_{-1}^{0}\int_{0}^{\sqrt{1-x^2}} e^{x^2+y^2}\,dy\,dx = \int_{0}^{\frac{\pi}{2}}\int_{0}^{1} r e^{r^2}\,dr\,d\theta$$

Example 2. Find the area inside $x^2+y^2=1$.

This is a silly problem. The area inside a circle of radius r is πr^2. Our circle has radius 1; so the area inside it is $\boxed{\pi}$.

To find this answer in rectangular coordinates is a nuisance. I drew the picture on the next page (look at the second picture).

$$\text{area} = \int_{-1}^{1}\int_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dy\,dx = \int_{-1}^{1} y\Big|_{-\sqrt{1-x^2}}^{\sqrt{1-x^2}} dx = \int_{-1}^{1} \sqrt{1-x^2}-(-\sqrt{1-x^2})dx = 2\int_{-1}^{1}\sqrt{1-x^2}dx$$

To do this integral, I have to make a Trig substitution. ugh.

On the other hand, the integral is easy in polar coordinates. Now look at the top picture on the next page.

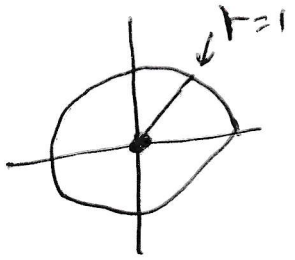
$$\text{area} = \int_0^{2\pi}\int_0^1 r\,dr\,d\theta = \int_0^{2\pi}\frac{r^2}{2}\Big|_0^1 d\theta = \int_0^{2\pi}\frac{1}{2}\,d\theta = \frac{1}{2}\theta\Big|_0^{2\pi} = \boxed{\pi}.$$

There was no effort involved and we got the right answer.

The picture is on the next page.

15.4 Second Example

Find the area inside $x^2+y^2=1$

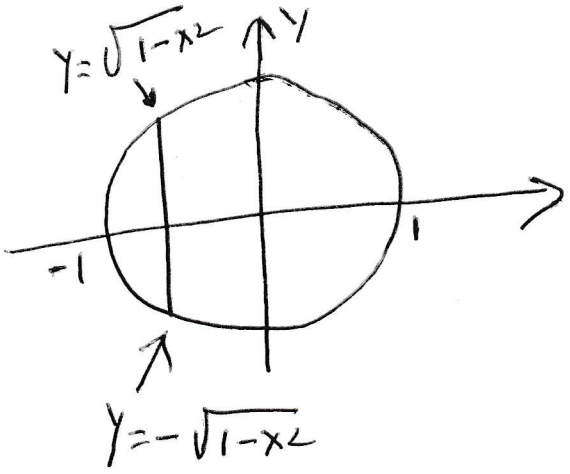


For each fixed θ between 0 and 2π, r goes from 0 to 1.

$$\text{The area} = \int_0^{2\pi} \int_0^1 r\,dr\,d\theta$$

In rectangular coordinates one might say for each x between -1 and 1, y goes from

$$-\sqrt{1-x^2} \quad \text{to} \quad +\sqrt{1-x^2}$$



Example 3. This is problem 19 from section 15.4. Compute

$$\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}}\, dx\, dy.$$

I do not know if I can compute $\int e^{\sqrt{x^2+y^2}}\, dx$. If I were trying to do it, I would first let $x = y\tan\theta$. Then I would calculate that $dx = y\sec^2\theta d\theta$. I am not eager to see if I could finish the problem using this approach.

I do notice that $x = \sqrt{(\ln 2)^2 - y^2}$ is just the part of the circle

$$x^2 + y^2 = (\ln 2)^2$$

where x is positive. This is the right half of the circle. The circle in question has radius $\ln 2$. For each fixed y between $y = 0$ and the top of the circle (that is, $y = \ln 2$), x goes from $x = 0$ (that is the y-axis) until $x = \sqrt{(\ln 2)^2 - y^2}$, (that is the right boundary of the circle.) The integral is taken over the first quadrant of a circle of radius $\ln 2$. There is a picture on the next page.

In polar coordinates, one fixes θ between $\theta = 0$ and $\theta = \frac{\pi}{2}$. For each such θ, r goes from 0 to $\ln 2$. Of course, $\sqrt{x^2+y^2} = r$ and the original integral becomes

$$\int_0^{\frac{\pi}{2}} \int_0^{\ln 2} re^r\, dr\, d\theta.$$

Use integration by parts to integrate $\int re^r dr$. Recall that integration by parts is

$$\int udv = uv - \int vdu.$$

We take $u = r$ and $dv = e^r\, dr$. We compute $du = dr$ and $v = e^r$. Thus,

$$\int re^r\, dr = re^r - \int e^r\, dr = re^r - e^r.$$

(It is easy to check in one's head that the derivative of $re^r - e^r$ really is re^r.) Our integral is

$$\int_0^{\frac{\pi}{2}} (re^r - e^r)\Big|_0^{\ln 2} d\theta = \int_0^{\frac{\pi}{2}} (2\ln 2 - 2 + 1)\, d\theta = (2\ln 2 - 1)\theta\Big|_0^{\frac{\pi}{2}} = \boxed{\frac{(2\ln 2 - 1)\pi}{2}}.$$

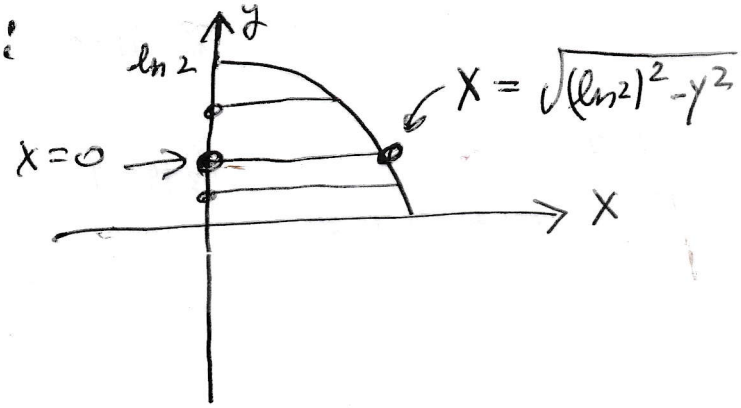
The picture is on the next page.

15.4 Third Example

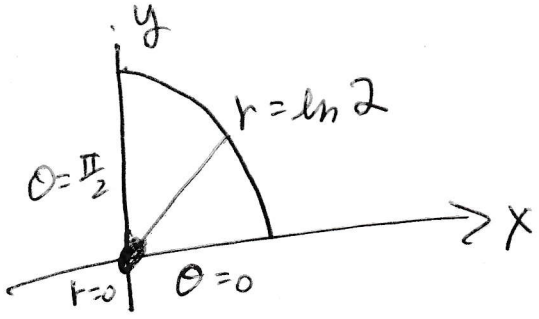
$$\int_0^{\ln 2} \int_0^{\sqrt{(\ln 2)^2 - y^2}} e^{\sqrt{x^2+y^2}} \, dx \, dy$$

$x = \sqrt{(\ln 2)^2 - y^2}$ is part of the circle $x^2 + y^2 = (\ln 2)^2$. This circle has radius $\ln 2$

The original region: For each fixed y between 0 and $\ln 2$, x goes from the y-axis (i.e. $x=0$) to the right part of the circle:



In polar coordinates



For each fixed θ with $0 \le \theta \le \frac{\pi}{2}$ r goes from 0 to $r = \ln 2$

19. Section 15.5. Triple integrals.

The triple integral, the double integral, the definite integral of first semester calculus, and the line integral of chapter 16 (alas, we will not get there) are all defined the same way. See the beginning of section 17 on page 100 for the definitions of the definite integral and the double integral. I will now define the triple integral.

The definition of the triple integral. Let S be a solid in 3-space and let $f(x,y,z)$ be a function defined on S. Use planes parallel to the xy-plane, the xz-plane, and the yz-plane to partition S into many small rectangular, three-dimensional boxes. Number the boxes from 1 to N. For each integer i with $1 \leq i \leq N$, let (x_i^*, y_i^*, z_i^*) be a point in box_i and let Δ_i be the volume of box_i. Then

$$\iiint_S f(x,y,z)\,dV = \lim_{\text{all } \Delta_i \text{ go to zero}} \sum_{i=1}^{N} f(x_i^*, y_i^*, z_i^*)\Delta_i.$$

Application 1 of the triple integral. If $f(x,y,z) = 1$ for all points (x,y,z), then

$$\iiint_S dV = \text{ the volume of } S.$$

I hope this is obvious. We chopped S into a bunch of little boxes. We found the volume of each box. We multiplied the volume of each box by one. We added all of the products

$$1 \text{ times the volume of box}_i.$$

Then we took the limit as we made more and more boxes allowing the size of each box to become arbitrarily small. Any given one of these sums is exactly the volume of some approximation of S. As the boxes get smaller our approximation gets better.

What we gain here is ease of notation rather than amazing computational ability. We can find volumes using double or single integrals but we have to work hard to describe what we are doing. It is hard to imagine anything easier to write than

$$\iiint_S dV = \text{ the volume of } S.$$

Of course, to actually figure out how to calculate $\iiint_S dV$, we will have to think about what S actually looks like.

Application 2 of the triple integral. If $\delta(x,y,z)$ is the mass density the solid S at the point (x,y,z), then

$$\iiint_S \delta(x,y,x)\,dV = \text{ the mass of } S.$$

Our solid is more dense at some spots than at other spots. If we want to approximate the total mass of the solid, we would probably chop the solid into many pieces, pretend the density is constant on each piece, calculate the pretend-constant density times volume to compute the mass of each piece, then add up the approximate masses of the pieces.

If we wanted a better approximation, we would make more pieces and insist that all pieces are smaller than they used to be.

To set up a triple integral: Pick a representative cross-section and:

• Suppose R is a representative cross section of the solid S with R parallel to the xy-plane. Let $z = t(x,y)$ be the top of S above the point $(x,y) \in R$ and $z = b(x,y)$ be the bottom of S below the point $(x,y) \in R$. Then[28]

$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{b(x,y)}^{t(x,y)} f(x,y,z)\,dz \right) dA.$$

See the top picture on the next page.

• Suppose R is a representative cross section of the solid S with R parallel to the xz-plane. Let $y = r(x,z)$ be the right-most point of S to the right of the point $(x,z) \in R$ and $y = \ell(x,z)$ be the left-most point of S to the left of the point $(x,z) \in R$. Then

$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{\ell(x,z)}^{r(x,z)} f(x,y,z)\,dy \right) dA.$$

See the middle picture on the next page.

• Suppose R is a representative cross section of the solid S with R parallel to the yz-plane. Let $x = fr(y,z)$ be the front point of S in front of the point $(y,z) \in R$ and $x = b(y,z)$ be the back point of S to behind the point $(y,z) \in R$. Then

$$\iiint_S f(x,y,z)\,dV = \iint_R \left(\int_{b(y,z)}^{fr(y,z)} f(x,y,z)\,dx \right) dA.$$

See the bottom picture on the next page – but I had enough trouble drawing the front, I did not even try to draw the back.

Please notice that the inner integral is always

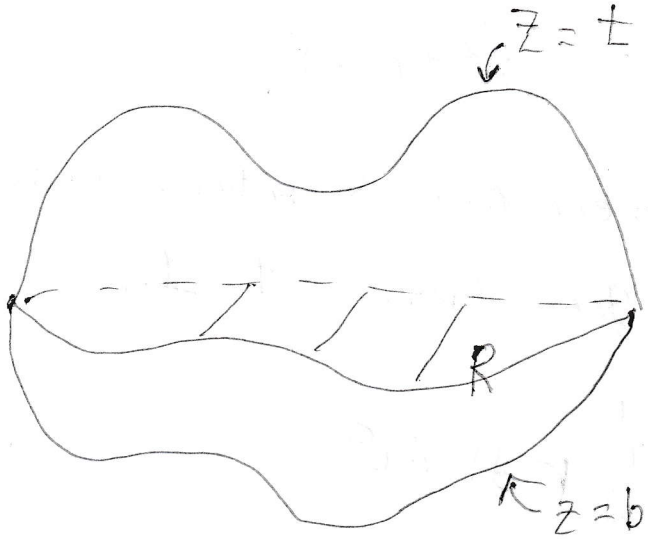
$$\int_{\text{the smallest value of the variable}}^{\text{the largest value of the variable}}.$$

That is why the integral is bottom to top, left to right, and back to front.

As promised, the next page consists of three pictures.

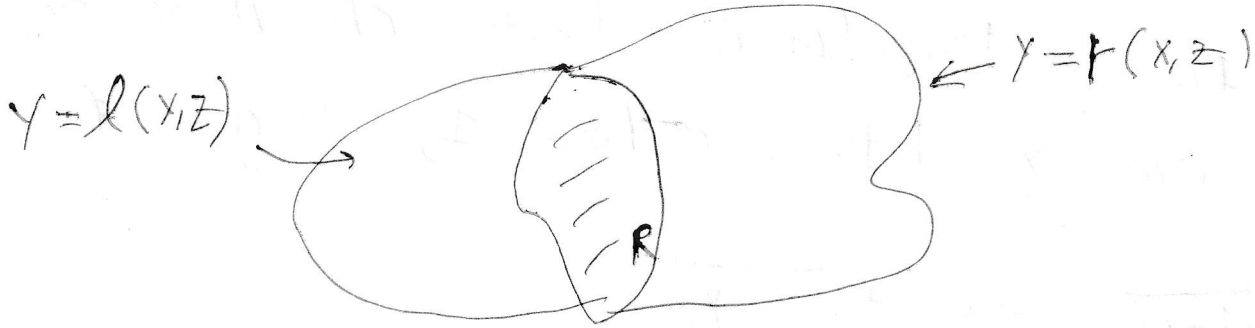
[28] Of course, nobody ever puts parentheses in an iterated integral. I only did it here to draw your attention to how I am doing things.

Set up a Triple integral



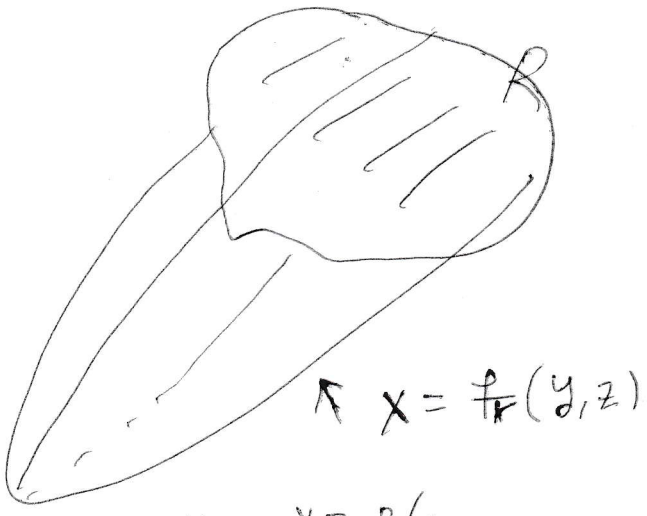
R is parallel to the XY Plane

for $(x,y) \in R$



R is parallel to the XZ plane

for $(x,z) \in R$



for $(y,z) \in R$

R is Parallel to the YZ plane

Example 1. This is problem 42 from 15.5. Compute

$$\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz.$$

ACK!! None of us can compute $\int ze^{zy^2}\,dy$. (If you think you can do it, you are wrong. Please take the time to figure out what you are doing wrong.) This is yet another trick problem. I visualize the problem as $\int_0^1 (\text{some double integral})dz$. The given problem integrates by filling up the region for the double integral by using vertical lines. I wonder if switching to horizontal lines will help. The given integral

$$\int_0^1 \int_{x^2}^1 12xze^{zy^2}\,dy\,dx$$

fills the region by thinking for each fixed x with x between 0 and 1, y goes from the parabola $y = x^2$ to the horizontal line $y = 1$. If we use horizontal lines to fill this region, then we say for each fixed y with y between 0 and 1, x goes from the y-axis (which is $x = 0$) to the parabola (which is $x = \sqrt{y}$). You must draw a picture to see this. My picture is on the next page.

At any rate,

$$\int_0^1 \int_0^1 \int_{x^2}^1 12xze^{zy^2}\,dy\,dx\,dz = \int_0^1 \int_0^1 \int_0^{\sqrt{y}} 12xze^{zy^2}\,dx\,dy\,dz.$$

Now at least we can do the first integral!

$$\text{the problem} = \int_0^1 \int_0^1 12\left(\frac{x^2}{2}\right) ze^{zy^2}\Big|_0^{\sqrt{y}}\,dy\,dz$$

$$= \int_0^1 \int_0^1 6yze^{zy^2}\,dy\,dz.$$

We can do the second integral! Let $u = zy^2$. Compute $du = 2zydy$.

$$\int 6yze^{zy^2}dy = \int 3e^u du = 3e^u + C = 3e^{zy^2} + C.$$

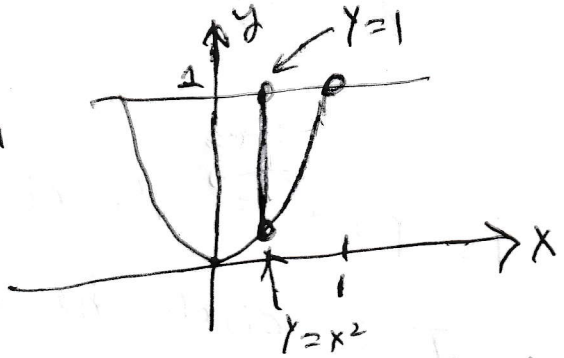
Check in your head that the partial derivative of $3e^{zy^2}$ with respect to y is $6yze^{zy^2}$. So

$$\text{the problem} = \int_0^1 3e^{zy^2}\Big|_0^1 dz = \int_0^1 (3e^z - 3)\,dz = (3e^z - 3z)\Big|_0^1 = (3e-3)-(3-0) = \boxed{3e-6}.$$

As promised, the picture is on the next page.

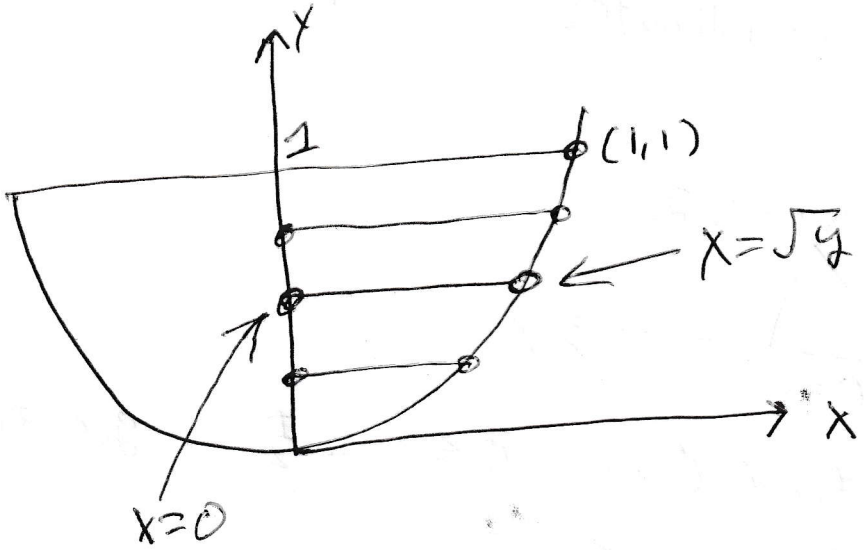
Section 15.5 Example 1

$\int_0^1 \int_{x^2}^1 \; dy\, dx$ represents integration over the region



For each fixed x with $0 \le x \le 1$, y goes from the parabola $y = x^2$ to the horizontal line $y = 1$.

We can fill this region using horizontal lines, for each fixed y with $0 \le y \le 1$, x goes from the y-axis (which is $x = 0$) to the parabola which is $x = \sqrt{y}$



Example 2. This is problem 15.5 number 36. Find the volume of the region bounded in the back by the plane $x = 0$, on the front and sides by the parabolic cylinder $x = 1 - y^2$, on the top by the paraboloid $z = x^2 + y^2$, and on the bottom by the xy-plane.

I drew what I can draw on the next page. The volume is

$$\iint_{\text{base}} \int_{\text{bottom}}^{\text{top}} dz\,dA = \iint_{\text{base}} \int_0^{x^2+y^2} dz\,dA = \iint_{\text{base}} z\Big|_0^{x^2+y^2} dA = \iint_{\text{base}} (x^2+y^2)\,dA.$$

You have to draw the base to set up the double integral. (My picture continues to be on the next page.) I fill the region with horizontal lines. For each fixed y with y between -1 and 1, x goes from the y-axis (which is $y = 0$) to the parabola. The volume is equal to

$$\int_{-1}^{1} \int_0^{1-y^2} (x^2+y^2)\,dx\,dy = \int_{-1}^{1} \left(\frac{x^3}{3} + xy^2 \right) \Big|_0^{1-y^2} dy = \int_{-1}^{1} \left(\frac{(1-y^2)^3}{3} + (1-y^2)y^2 \right) dy$$

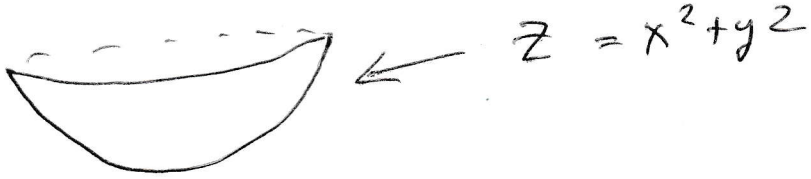
$$= \int_{-1}^{1} \left(\frac{1-3y^2+3y^4-y^6}{3} + y^2 - y^4 \right) dy = \frac{1}{3} \int_{-1}^{1} (1-3y^2+3y^4-y^6+3y^2-3y^4)\,dy$$

$$= \frac{1}{3} \int_{-1}^{1} (1-y^6)\,dy = \frac{1}{3} \left(y - \frac{y^7}{7} \right) \Big|_{-1}^{1} = \frac{1}{3} \left(1 - \frac{1}{7} - \left(-1 + \frac{1}{7} \right) \right) = \frac{1}{3} \frac{2(6)}{7} = \boxed{\frac{4}{7}}.$$

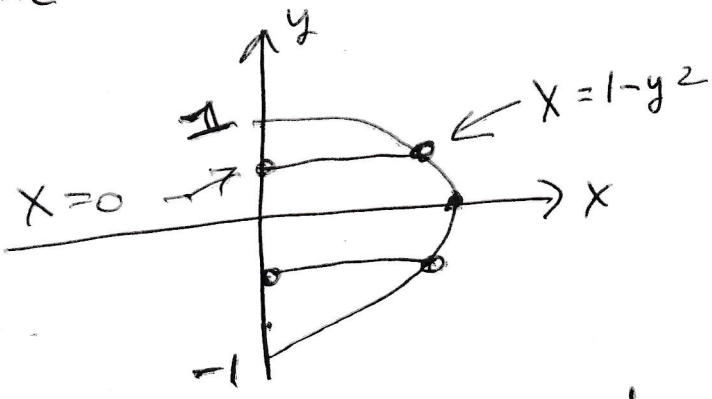
As promised, there is a picture on the next page.

Section 15.5 Example 2

The top looks like



The bottom looks like



$$\text{Vol} = \iint_{\text{base}} \text{top}\, dA = \int_{-1}^{1} \int_{0}^{1-y^2} (x^2 + y^2)\, dx\, dy$$

20. **Section 15.7. Triple integrals in cylindrical and spherical coordinates.**

The purpose of this lecture is to make some integrals that would be hard or impossible easier.

You already know cylindrical coordinates; they are r, θ, and z. Of course, $dx\,dy\,dz$ in some order becomes $r\,dz\,dr\,d\theta$ in some order. We will do some problems in cylindrical coordinates, but there is no new lecture about them.

Spherical coordinates are probably new to you. The spherical coordinates are ρ, ϕ, and θ. The coordinate ρ is the distance from the origin to the point. (So $\rho = 2$ is the sphere of radius 2 with center at the origin.) The coordinate ϕ is the angle from the positive z-axis to the line segment joining the point to the origin. (So, $\phi = 0$ is the positive z-axis; $\phi = \frac{\pi}{4}$ is cone; $\phi = \frac{\pi}{2}$ is the xy-plane, $\phi = \frac{3\pi}{4}$ is an up-side-down cone, $\phi = \pi$ is the negative z-axis.) In fact, ϕ only makes sense for $0 \leq \phi \leq \pi$. Of course θ is the same θ as in polar coordinates. I drew pictures of ϕ equal to various constants on the next page.

We know how to convert back and forth between rectangular coordinates and cylindrical coordinates. To convert back and forth between cylindrical coordinates and spherical coordinates draw a right triangle as shown on the next page. Label the top side (which is the opposite) r; label the left side (which is the adjacent) z; label the hypotenuse ρ; and label the angle ϕ. I drew this triangle on the next page.

We see that

$$z^2 + r^2 = \rho^2$$
$$z = \rho\cos\phi$$
$$r = \rho\sin\phi$$

It follows that

$$dx\,dy\,dz = \rho^2\sin\phi\,d\rho\,d\phi\,d\theta.$$

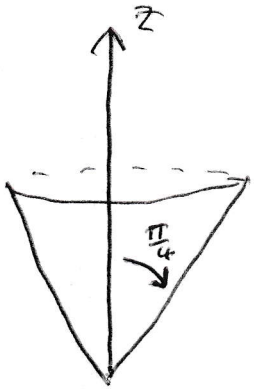
To give more of the story; **if you care**. The conversion between rectangular coordinates and spherical coordinates is given by

$$x = r\cos\theta = \rho\sin\phi\cos\theta$$
$$y = r\sin\theta = \rho\sin\phi\sin\theta$$
$$z = \rho\cos\phi$$

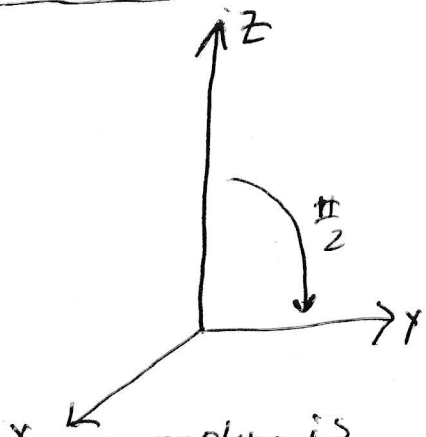
I will write more about this conversion from rectangular coordinates to spherical coordinates after the picture.

The next page has the promised picture.

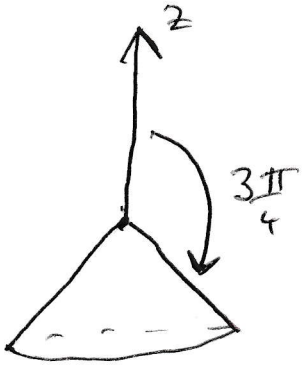
Spherical Coordinates



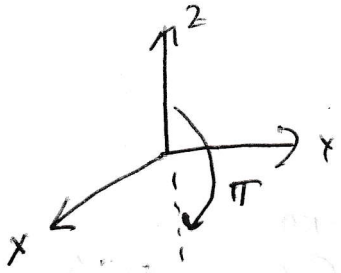
This cone is $\varphi = \frac{\pi}{4}$



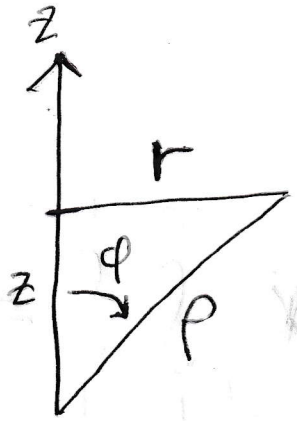
The xy plane is $\varphi = \frac{\pi}{2}$



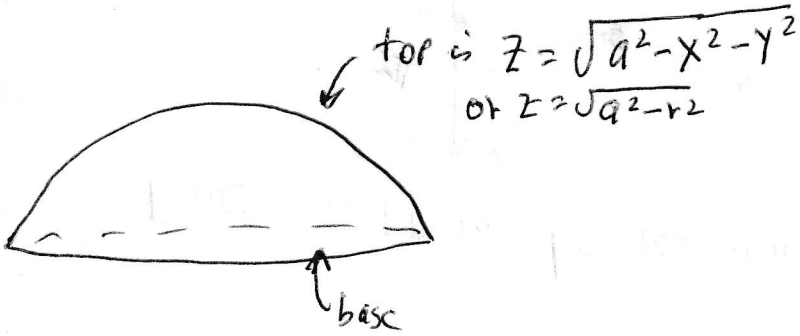
This cone is $\varphi = \frac{3\pi}{4}$



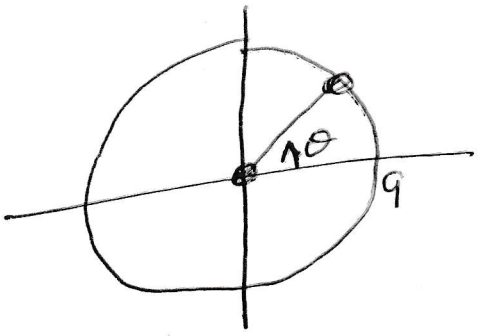
The negative z-axis is $\varphi = \pi$



For example 1



The base is



for each θ with $0 \le \theta \le 2\pi$

r goes from 0 to a

The change of coordinates formula for integrals is given as formula (18.0.1) on page 125:

$$dx\,dy\,dz = \left|\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}\right| d\rho\, d\phi\, d\theta.$$

If you make the calculation, you will get

$$dx\,dy\,dz = \rho^2 \sin\phi\, d\rho\, d\phi\, d\theta.$$

Example 1. Find the volume inside $x^2+y^2+z^2=a^2$ and above the xy-plane. Of course, the problem is silly. The equation $x^2+y^2+z^2=a^2$ is a sphere of radius a. The volume of a sphere is $\frac{4}{3}\pi(\text{radius})^3$. Our region is one half of the sphere and the radius is a. So the answer to our problem is $\boxed{\dfrac{2\pi a^3}{3}}$. Nonetheless, it surely is informative to see how the various methods of integration work on this problem.

Rectangular coordinates. Forget it. It is much too hard.

Cylindrical coordinates. I put the picture on the PREVIOUS page. The volume is

$$\iint_{\text{base}} \int_0^{\text{top}} dz\,dx\,dy = \iint_{\text{base}} \sqrt{a^2-x^2-y^2}\,dx\,dy = \int_0^{2\pi}\int_0^a r\sqrt{a^2-r^2}\,dr\,d\theta.$$

To compute $\int r\sqrt{a^2-r^2}\,dr$, let $u=a^2-r^2$. Calculate $du=-2r\,dr$; so

$$\int r\sqrt{a^2-r^2}\,dr = -\frac{1}{2}\int u^{1/2}du = -\frac{1}{2}\frac{2}{3}u^{3/2}+C = \frac{-1}{3}(a^2-r^2)^{3/2}+C. \tag{20.0.1}$$

A quick check shows that this is correct. So, the volume is

$$= \frac{-1}{3}\int_0^{2\pi}(a^2-r^2)^{3/2}\Big|_0^a d\theta = \frac{1}{3}\int_0^{2\pi} a^3\,d\theta = \frac{1}{3}a^3\theta\Big|_0^{2\pi} = \boxed{\frac{2\pi a^3}{3}},$$

as expected.

Spherical coordinates. For each fixed θ and ϕ, ρ goes from 0 to a. Of course, ϕ can be any number between 0 and $\frac{\pi}{2}$, and $0\le\theta\le 2\pi$. Thus, the volume is

$$\int_0^{2\pi}\int_0^{\pi/2}\int_0^a \rho^2\sin\phi\,d\rho\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2}\frac{\rho^3}{3}\Big|_0^a \sin\phi\,d\phi\,d\theta = \int_0^{2\pi}\int_0^{\pi/2}\frac{a^3}{3}\sin\phi\,d\phi\,d\theta$$

$$= \int_0^{2\pi} -\frac{a^3}{3}\cos\phi\Big|_0^{\pi/2} d\theta = \int_0^{2\pi} -\frac{a^3}{3}(0-1)\,d\theta = \frac{a^3}{3}\theta\Big|_0^{2\pi} = \boxed{\frac{2\pi a^3}{3}}.$$

Example 2. Find the volume of the solid below $x^2+y^2+z^2=1$ and above the top part of $x^2+y^2=z^2$.

Again, the problem is too hard in rectangular coordinates.

Spherical coordinates. This problem is easiest in spherical coordinates. Notice that the sphere $x^2+y^2+z^2=1$ is merely $\rho=1$ spherical coordinates. Notice also that the cone $x^2+y^2=z^2$ is $r^2=z^2$ or $z=\pm r$. The problem says that we are interested in the top part of $z=\pm r$, so we are interested in $z=r$. Look at the picture that relates r, z, ρ, and ϕ. When $r=z$, the triangle is an isosceles right triangle and $\phi=\frac{\pi}{4}$. (See the picture on the last page.)

For each fixed θ (with $0\le\theta\le 2\pi$) and each fixed ϕ with $0\le\phi\le\frac{\pi}{4}$, ρ goes from 0 to 1. The volume of the solid is

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^1 \rho^2\sin\phi\,d\rho\,d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi/4}\frac{\rho^3}{3}\sin\phi\Big|_0^1 d\phi\,d\theta=\int_0^{2\pi}\int_0^{\pi/4}\frac{1}{3}\sin\phi\,d\phi\,d\theta$$
$$=\int_0^{2\pi}\frac{1}{3}(-\cos\phi)\Big|_0^{\pi/4}d\theta=\int_0^{2\pi}\frac{1}{3}\left(-\frac{\sqrt{2}}{2}+1\right)d\theta=\frac{2-\sqrt{2}}{6}\theta\Big|_0^{2\pi}=\boxed{\frac{(2-\sqrt{2})\pi}{3}}.$$

Cylindrical coordinates. We can do the problem in cylindrical coordinates. It is a little harder; but it does work. At any rate, the volume is

$$\iint_{\text{the fattest cross section}}\int_{\text{bottom}}^{\text{top}} dz\,dA$$

The fattest cross section is the intersection of the sphere ($x^2+y^2+z^2=1$) and the cone ($x^2+y^2=z^2$); namely, the intersection of $r^2+z^2=1$ and $r^2=z^2$. The intersection is the circle $r^2+r^2=1$ with $z=r$. The intersection is $r=\frac{1}{\sqrt{2}}$ and $z=\frac{1}{\sqrt{2}}$. Of course, the top is $z=\sqrt{1-r^2}$ and the bottom is $z=r$. (These graphs are labeled in the bottom picture on the last page; but they are the same graphs as appear in the top picture with x^2+y^2 now written as r^2.)

The volume is

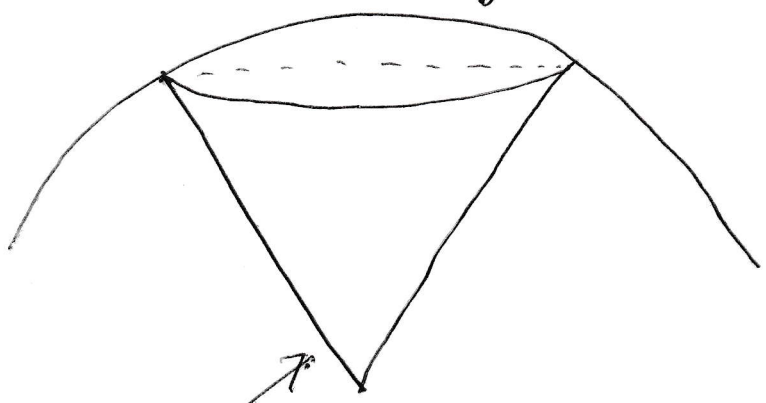
$$\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\int_r^{\sqrt{1-r^2}} r\,dz\,dr\,d\theta=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}} rz\Big|_r^{\sqrt{1-r^2}}dr\,d\theta=\int_0^{2\pi}\int_0^{\frac{1}{\sqrt{2}}}\left(r\sqrt{1-r^2}-r^2\right)dr\,d\theta.$$

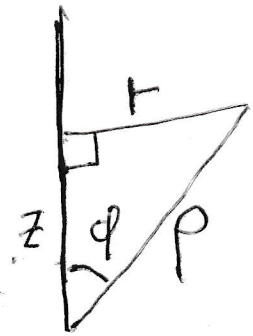
Look at (20.0.1) on page 141 to see that $\int r\sqrt{1-r^2}\,dr=\frac{-1}{3}(1-r^2)^{3/2}+C$. The volume is

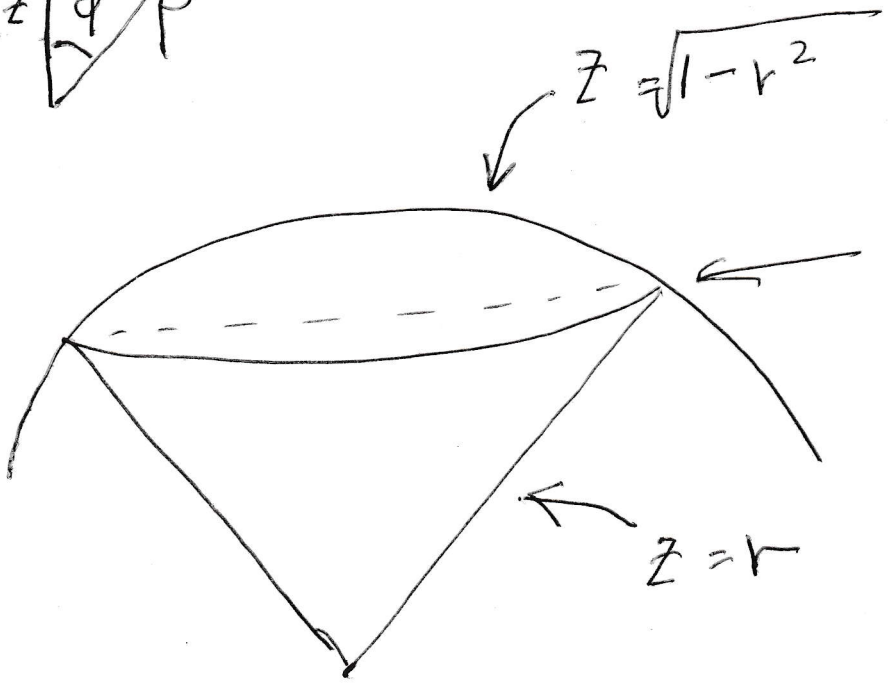
$$\int_0^{2\pi}\left(\frac{-1}{3}(1-r^2)^{3/2}-\frac{r^3}{3}\right)\Big|_0^{\frac{1}{\sqrt{2}}}d\theta$$
$$=\int_0^{2\pi}\left[\left(\frac{-1}{3}\left(1-\frac{1}{2}\right)^{3/2}-\frac{1}{3(2\sqrt{2})}\right)-\left(\frac{-1}{3}(1-0)^{3/2}-\frac{0^3}{3}\right)\right]d\theta$$
$$=\int_0^{2\pi}\left[\left(\frac{-1}{(3)2\sqrt{2}}-\frac{1}{3(2\sqrt{2})}\right)+\frac{1}{3}\right]d\theta\int_0^{2\pi}\left[\frac{1}{3}-\frac{1}{3\sqrt{2}}\right]d\theta$$
$$=\left[\frac{1}{3}-\frac{1}{3\sqrt{2}}\right]\theta\Big|_0^{2\pi}=\boxed{\frac{1}{3}\left(1-\frac{\sqrt{2}}{2}\right)2\pi}.$$

We are very pleased to see that we got the same answer as we got when we did the problem the other way. There are pictures on the next page.

15.7 Example 2







21. **Substitution in double and triple integrals.**

Fact 21.1. *If $x = x(u,v)$ and $y = y(u,v)$, then*

$$\iint_{R \text{ written with respect to } x,y} f(x,y) \quad \underbrace{dx\,dy}_{\text{in some order}}$$

$$= \iint_{R \text{ written with respect to } u,v} f(x(u,v),y(u,v))\left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| \quad \underbrace{du\,dv}_{\text{in some order}} .$$

(*The big* $\left|\ \right|$ *means determiant and the little* $|\ |$ *means absolute value.*)

Examples 21.2. (a) In first semester if you are supposed to compute $\int_I f(x)dx$, but you decide to use a different variable u, you figure out what $x = x(u)$, and you compute

$$\int_{I \text{ written in terms of } u} f(x(u))\frac{dx}{du}du.$$

(There is no absolute value in first semester calculus because when you change I from x-coordinates to u-coordinates it is easy to just integrate from big to little in u coordinates. That is $\int_1^2 \sqrt{5-x}dx = -\int_4^3 \sqrt{u}du$, when $u = 5-x$. It is harder to know exactly what to switch when one does a substitution in double or triple integrals.)

(b) The switch to polar coordinates from rectangular coordinates is

$$\iint_R f(x,y)dx\,dy = \iint_{R \text{ in polar coordinates}} f(r\cos\theta, r\sin\theta)d\theta\,dr$$

Of course,

$$\left|\begin{vmatrix} \frac{\partial x}{\partial \theta} & \frac{\partial x}{\partial r} \\ \frac{\partial y}{\partial \theta} & \frac{\partial y}{\partial r} \end{vmatrix}\right| = r.$$

(c) The switch to spherical coordinates from rectangular coordinates is

$$\iiint_{\mathscr{S}} f(x,y,z) \ \underbrace{dx\,dy\,dz}_{\text{in some order}}$$

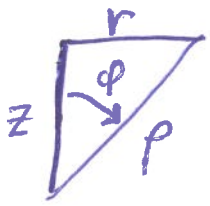
$$= \iiint_{\mathscr{S} \text{ in spherical coordinates}} f(\rho\sin\phi\cos\theta, \rho\sin\phi\sin\theta, \rho\cos\phi)\left|\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}\right| \ \underbrace{d\rho\,d\phi\,d\theta}_{\text{in some order}}$$

The Jacobian

$$\left|\begin{vmatrix} \frac{\partial x}{\partial \rho} & \frac{\partial x}{\partial \phi} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial \rho} & \frac{\partial y}{\partial \phi} & \frac{\partial y}{\partial \theta} \\ \frac{\partial z}{\partial \rho} & \frac{\partial z}{\partial \phi} & \frac{\partial z}{\partial \theta} \end{vmatrix}\right|$$

turns out to be $\rho^2\sin\phi$. A picture explaining the conversion of rectangular coordinates to spherical coordinates is on the next page.

Convert Rectangular coordinates to Spherical coordinates



$z = \rho \cos\varphi$

$r = \rho \sin\varphi$

$$x = r\cos\theta = \rho \sin\varphi \cos\theta$$

$$y = r\sin\theta = \rho \sin\varphi \sin\theta$$

$$z = \rho \cos\varphi$$

Example 21.3. Find the area inside $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.

Let $u=\frac{x}{a}$ and $v=\frac{y}{b}$. In u,v coordinates the region becomes the area inside $u^2+v^2=1$. This is a circle of radius 1. The inside this circle is $\pi(\text{radius})^2=\pi$.

Let R be the area inside $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is

$$\iint_R 1 \underbrace{dx\,dy}_{\text{in some order}} = \iint_{R \text{ in } uv\text{-coordinates}} \left|\begin{vmatrix} \frac{dx}{du} & \frac{dx}{dv} \\ \frac{dy}{du} & \frac{dy}{dv} \end{vmatrix}\right| \underbrace{du\,dv}_{\text{in some order}}$$

Write $au=x$ and $bv=y$ to compute the Jacobian.

$$= \iint_{R \text{ in } uv\text{-coordinates}} \left|\begin{vmatrix} a & 0 \\ 0 & b \end{vmatrix}\right| \underbrace{du\,dv}_{\text{in some order}}$$

$$= \iint_{\text{the circle of radius 1 with center } (0,0)} ab\,du\,dv = \boxed{ab\pi}.$$

Example 21.4. Use the transformation

$$u=x+2y \quad \text{and} \quad v=x-y$$

to calculate

$$\int_0^{2/3}\int_y^{2-2y} (x+2y)e^{y-x}\,dx\,dy.$$

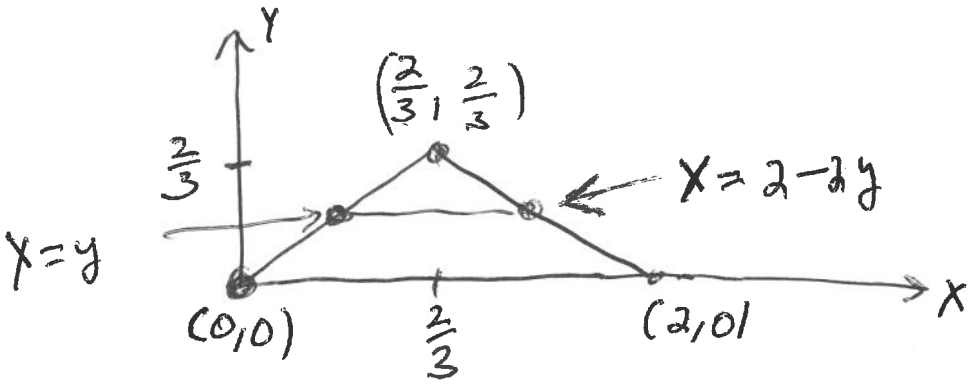
First we solve for x and y, so that we can calculate the Jacobian

$$\left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right|$$

Observe that $u-v=3y$; so $\frac{u-v}{3}=y$. Observe also that $u+2v=3x$; so $\frac{u+2v}{3}=x$. Now we compute

$$\left|\begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}\right| = \left|\begin{vmatrix} \frac{1}{3} & \frac{2}{3} \\ \frac{1}{3} & -\frac{1}{3} \end{vmatrix}\right| = \left|-\tfrac{3}{9}\right| = \tfrac{1}{3}.$$

We see that the original integral is taken over the following region. For each fixed y, with $0\le y\le\frac{2}{3}$, x goes from y to $2-2y$. So the original region is the triangle with boundary $x=y$, $x=2-2y$, and $y=0$. The vertices are $(0,0)$, $(\frac{2}{3},\frac{2}{3})$, and $(2,0)$:



The region of the original integral

(The point $(\frac{2}{3},\frac{2}{3})$ is the intersection of the lines $x=y$ and $x=2-2y$.)

Observe that $x=y$ becomes $\frac{u+2v}{3}=\frac{u-v}{3}$ or $u+2v=u-v$ or $v=0$ in the new variables u,v.

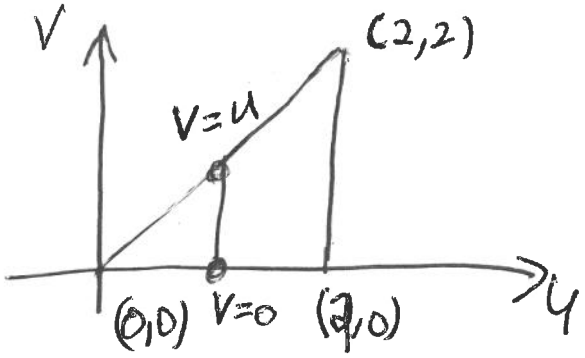
Observe that $x=2-2y$ becomes $\frac{u+2v}{3}=2-2\frac{u-v}{3}$ or $u+2v=6-2(u-v)$ or $3u=6$ or $u=2$ in the new variables u,v.

Observe also, that $y=0$ becomes $\frac{u-v}{3}=0$ or $u=v$ in the new variables u,v.

The old vertices (x,y) equals $(0,0)$, $(\frac{2}{3},\frac{2}{3})$, and $(2,0)$ are sent to (u,v) equal $(0,0)$, $(2,0)$, and $(2,2)$, respectively.

In (u,v) coordinates, we integrate over:

The original region in (U,V) coordinates



To integrate over the domain in uv-coordinates, first fix u with $0 \leq u \leq 2$. For each fixed u, v goes from 0 to u.

The original integral is equal to

$$\begin{aligned}
\frac{1}{3}\int_0^2 \int_0^u ue^{-v}\,dv\,du &= \frac{1}{3}\int_0^2 -ue^{-v}\Big|_0^u du \\
&= \frac{1}{3}\int_0^2 (-ue^{-u}+u)\,du \\
&= \frac{1}{3}(ue^{-u}+e^{-u}+\frac{u^2}{2})\Big|_0^2 \qquad \text{See (21.4.1).} \\
&= \frac{1}{3}(2e^{-2}+e^{-2}+2-1) \\
&= \boxed{\frac{1}{3}(3e^{-2}+1).}
\end{aligned}$$

(21.4.1) We used integration by parts to see that $\int -ze^{-z}dz = ze^{-z}+e^{-z}+C$.