15.7, number 79: Find the volume of the solid region enclosed by the cylinder $x^2 + y^2 = 4$ and the planes $z = 0$ and $y + z = 4$.

Answer: The volume is equal to the integral over the base of the top. The top is $z = 4 - y$. In polar coordinates the top is $z = 4 - r \sin\theta$. The base is everthing inside the circle $x^2 + y^2 = 4$ in the xy-plane. In polar coordinates that is: for each fixed θ with $0 \le \theta \le 2\pi$, r goes from $r = 0$ to to $r = 2$. Thus, the volume is

$$\begin{aligned}
&\int_0^{2\pi} \int_0^2 (4 - r\sin\theta) r \, dr \, d\theta \\
&= \int_0^{2\pi} \int_0^2 (4r - r^2 \sin\theta) \, dr \, d\theta \\
&= \int_0^{2\pi} (2r^2 - \tfrac{r^3}{3} \sin\theta)\Big|_0^2 \, d\theta \\
&= \int_0^{2\pi} (8 - \tfrac{8}{3} \sin\theta) \, d\theta \\
&= (8\theta + \tfrac{8}{3}\cos\theta)\Big|_0^{2\pi} \\
&= 16\pi + \tfrac{8}{3} - \tfrac{8}{3} = \boxed{16\pi}
\end{aligned}$$