15.7, number 31: Compute

$$\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2\theta + z^2) r \, d\theta \, dr \, dz.$$

Answer:

$$\begin{aligned}
& \textstyle\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \cos^2\theta + z^2) r \, d\theta \, dr \, dz \\
= & \textstyle\int_0^1 \int_0^{\sqrt{z}} \int_0^{2\pi} (r^2 \frac{1}{2}(1 + \cos(2\theta)) + z^2) r \, d\theta \, dr \, dz \\
= & \textstyle\int_0^1 \int_0^{\sqrt{z}} (r^2 \frac{1}{2}(\theta + \frac{\sin(2\theta)}{2}) + z^2\theta) \Big|_0^{2\pi} r \, dr \, dz \\
= & \textstyle\int_0^1 \int_0^{\sqrt{z}} (\pi r^3 + 2\pi z^2 r) \, dr \, dz \\
= & \textstyle\int_0^1 (\pi \frac{r^4}{4} + 2\pi z^2 \frac{r^2}{2}) \Big|_0^{\sqrt{z}} \, dz \\
= & \textstyle\int_0^1 (\pi \frac{z^2}{4} + 2\pi z^2 \frac{z}{2}) \, dz \\
= & (\pi \frac{z^3}{12} + 2\pi \frac{z^4}{8}) \Big|_0^1 = \boxed{\pi \frac{\pi}{3}}
\end{aligned}$$