15.5, number 33: Find the volume of the region between the planes

$$x + y + 2z = 2 \quad \text{and} \quad 2x + 2y + z = 4$$

in the first octant.

Answer: We drew the picture on the next page. From our drawing we read that the volume is equal to

$$\begin{aligned}
& \int_0^2 \int_0^{2-x} \left((4 - 2x - 2y) - \tfrac{2-x-y}{2}\right) dy\, dx \\
= & \int_0^2 \int_0^{2-x} \left(3 - \tfrac{3}{2}x - \tfrac{3}{2}y\right) dy\, dx \\
= & \int_0^2 \left(3y - \tfrac{3}{2}xy - \tfrac{3}{4}y^2\right)\Big|_0^{2-x} dx \\
= & \int_0^2 \left(3(2-x) - \tfrac{3}{2}x(2-x) - \tfrac{3}{4}(2-x)^2\right) dx \\
= & \int_0^2 \left(6 - 6x + \tfrac{3}{2}x^2 - \tfrac{3}{4}(2-x)^2\right) dx \\
= & \left(6x - 3x^2 + \tfrac{1}{2}x^3 + \tfrac{1}{4}(2-x)^3\right)\Big|_0^2
\end{aligned}$$

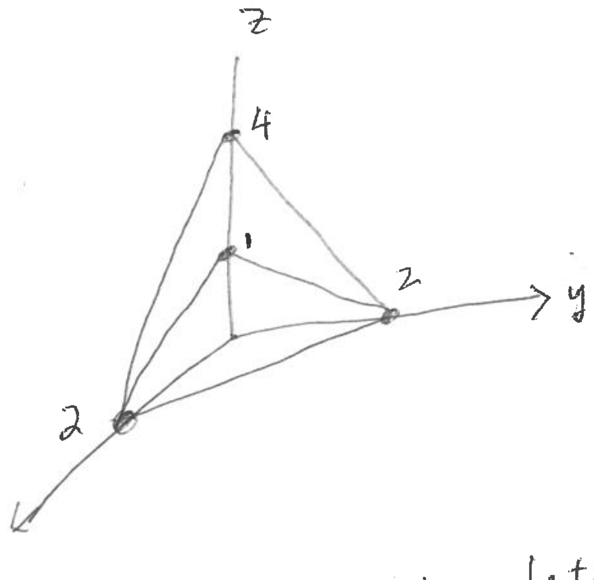
Of course, I made a substitution when I calculated the anti-derivative of $(2-x)^3$. I let $w = 2 - x$; so $dw = -dx$.

$$= 12 - 12 + 4 - 2 = \boxed{2}$$

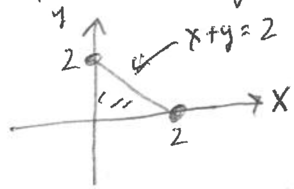
Picture 15.5 Number 33

$x+y+2z=2$ intersects the x-axis at $(2,0,0)$
intersects the y axis at $(0,2,0)$
intersects the z axis at $(0,0,1)$

$2x+2y+z=4$ intersects the x-axis at $(2,0,0)$
intersects the y-axis at $(0,2,0)$
intersects the z-axis $(0,0,4)$



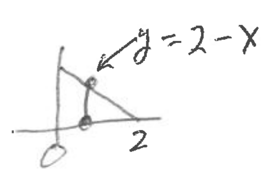
We want the volume of the region between the two planes. We integrate over



. We integrate

the value of z at (x,y) on the top plane − the value of z at (x,y) on the bottom plane

$$= (4-2x-2y) - \frac{2-x-y}{2}.$$



Fill the triangle with vertical lines

$$Vol = \int_0^2 \int_0^{2-x} \left((4-2x-2y) - \frac{2-x-y}{2}\right) dy\, dx$$