15.2, number 65: **Find the volume of the region bounded above by the paraboloid $z = x^2 + y^2$ and below by the triangle enclosed by the lines $y = x$, $x = 0$, and $x + y = 2$ in the xy-plane.**

Answer: The volume of a solid with base in the xy-plane is equal to the double integral over the base of the top. We drew the base on the next page. Look at the picture. We fill base up using vertical lines. For each fixed x with $0 \le x \le 1$, y goes from $y = x$ to $y = 2 - x$. The volume is equal to

$$\begin{aligned}
& \int_0^1 \int_x^{2-x} (x^2 + y^2)\, dy\, dx \\
= {} & \int_0^1 \left(x^2 y + \tfrac{y^3}{3} \right)\Big|_x^{2-x} dx \\
= {} & \int_0^1 \left(x^2(2-x) + \tfrac{(2-x)^3}{3} - (x^3 + \tfrac{x^3}{3}) \right) dx \\
= {} & \int_0^1 \left(2x^2 - \tfrac{7}{3}x^3 + \tfrac{(2-x)^3}{3} \right) dx
\end{aligned}$$

To integrate $(2-x)^3$, let $w = 2 - x$, so $dw = -dx$. It follows that the integral of $\frac{(2-x)^3}{3}$ is $-\frac{(2-x)^4}{12}$.

$$\begin{aligned}
= {} & \left(\tfrac{2x^3}{3} - \tfrac{7}{12}x^4 - \tfrac{(2-x)^4}{12} \right)\Big|_0^1 \\
= {} & \tfrac{2}{3} - \tfrac{7}{12} - \tfrac{1}{12} + \tfrac{16}{12} = \boxed{\tfrac{4}{3}}
\end{aligned}$$

Picture 15.2 Number 65

The triangle enclosed by $y=x$, $x=0$, and $x+y=2$ is