14.7, F20-e3-5: **Find the absolute extreme points of the function**

$$f(x, y) = x + y - xy,$$

which is defined on the closed triangle with vertices at $(0, 0)$**,** $(0, 2)$**, and** $(4, 0)$**.**

Answer:

We put a picture of the domain on the last page. We see that the boundary has three pieces. Eventually, we will look at f restricted to each of these three pieces. Eventually, also, we will look at f evaluated at each of the end points of the boundary.

First we look for interior points where both partial derivatives vanish. We compute $f_x = 1 - y$ and $f_y = 1 - x$. If $f_x = 0$ and $f_y = 0$ then $x = 1$ and $y = 1$. We will study $(1, 1)$ in our final step.

Now we look at f restricted to the vertical line $x = 0$, with $0 \le y \le 2$. This function is

$$f|_{x=0} = y.$$

We see that $\frac{d}{dy}(f|_{x=0}) = 1$, which is never zero. Thus, the extreme points of $f|_{x=0}$ occur at the end points $(0, 0)$ and $(0, 2)$. We already know to study these points in our final step.

Now we look at f restricted to the horizontal line $y = 0$, with $0 \le x \le 4$. This function is

$$f|_{y=0} = x.$$

We see that $\frac{d}{dx}(f|_{y=0}) = 1$, which is never zero. Thus, the extreme points of $f|_{y=0}$ occur at the end points $(0, 0)$ and $(4, 0)$. We already know to study these points in our final step.

Now we look at f restricted to the slanting line $y = -\frac{1}{2}x + 2$, with $0 \le x \le 4$. This function is

$$f|_{y=-\frac{1}{2}x+2} = x + (-\tfrac{1}{2}x + 2) - x(-\tfrac{1}{2}x + 2) = \tfrac{x^2}{2} - \tfrac{3}{2}x + 2.$$

We compute

$$\tfrac{d}{dx}(f|_{\text{slanting line}}) = x - \tfrac{3}{2}.$$

Thus, $\frac{d}{dx}(f|_{\text{slanting line}}) = 0$ when $x = \frac{3}{2}$ and $y = 2 - \frac{3}{4} = \frac{5}{4}$.

It is time for the final step. The extreme points of f on our domain occur at one of the points $(0, 0)$, $(0, 2)$, $(4, 0)$, $(1, 1)$, or $(\frac{3}{2}, \frac{5}{4})$. We evaluate f at these 5 points; the largest answer is the maximum. The smallest answer is the minimum.

$$\begin{aligned} f(0,0) &= 0 \\ f(0,2) &= 2 \\ f(4,0) &= 4 \\ f(1,1) &= 1 \\ f(\tfrac{3}{2}, \tfrac{5}{4}) &= \tfrac{7}{8} \end{aligned}$$

We conclude that $(4, 0, 4)$ is the maximum of f on our domain and $(0, 0, 0)$ is the minimum of f on our domain.